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Propagation of chaos in mean field networks of FitzHugh-Nagumo neurons.

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Abstract

In this article, we are interested in the behavior of a fully connected network of N neurons, where N tends to infinity. We assume that the neurons follow the stochastic FitzHugh-Nagumo model, whose specificity is the non-linearity with a cubic term. We prove a result of uniform in time propagation of chaos of this model in a mean-field framework. We also exhibit explicit bounds. We use a coupling method initially suggested by Eberle [9], and recently extended in [8], known as the reflection coupling. We simultaneously construct a solution of the N-particle system and N independent copies of the non-linear McKean-Vlasov limit in such a way that, considering an appropriate semi-metric that takes into account the various possible behaviors of the processes, the two solutions tend to get closer together as N increases, uniformly in time. The reflection coupling allows us to deal with the non-convexity of the underlying potential in the dynamics of the quantities defining our network, and show independence at the limit for the system in mean field interaction with sufficiently small Lipschitz continuous interactions.

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1 Introduction

1.1 Understanding the model

Understanding brain activity is both a complex and important challenge in current research. Of course, interests are plentiful: characterizing brain functions, unveiling structures and links between them, and understanding some phenomena such as cyclic heartbeat. A way of modeling this activity is by considering a very large number of individual neurons with interactions. Since the number of neurons in a human brain is around 10^{11} , and even "small" parts of the brain are thus constituted of a very large number of them, such a strategy can be considered coherent.

The main quantity we study is the membrane potential of the nerve cells: it can "easily" be observed and its modification characterizes a synapse (an interaction between

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neurons). Neurons regulate their electrical potential. In general, without interaction, the potential evolves with time but has quite small changes. Incoming potentials from other neurons are usually what make the neuron fire, *i.e.* send action potentials to other neurons. We will here focus on a homogeneous network of neurons and consider mean-field interactions. This way, each neuron will interact with every other one, as it can be the case in small regions of the brain. The parameters of the model will be considered the same for each neuron.

A classical model was introduced by Hodgkin and Huxley [11] using experimental data on the activity of the giant squid axon. It describes the ion exchanges K^+ , Na^+ , and Cl^- through the membrane and their effects on the potential. A simplification of this model is the FitzHugh-Nagumo model, which reduces the dimension: from a four-dimensional model (for one neuron) with the Hodgkin-Huxley equations, we obtain a two-dimensional model, thus yielding a compromise between biological accuracy and mathematical simplicity.

The deterministic FitzHugh-Nagumo model for one neuron (or one particle) is given by the following equations

$$\begin{cases} dX_t = (X_t - (X_t)^3 - C_t - \alpha)dt \\ dC_t = (\gamma X_t - C_t + \beta)dt, \end{cases}$$

where X is the membrane potential and C is a recovery variable, called the adaptation variable. The parameters γ and β are positive constants that determine the duration of excitation and the position of the equilibrium point of this system. Finally, $\alpha \in \mathbb{R}$ is the magnitude of a stimulus current (an entrance current in the system). Note that the variable C isn't a physical quantity, and is used to allow X to mimic the behavior of the potential. This variable C has linear dynamics and provides slower negative feedback.

This deterministic model has been largely studied. In Chapter 7 of [30], Thieullen describes the behavior of the solution of one deterministic FitzHugh-Nagumo system. She also extends the result in the case of a stochastic FitzHugh-Nagumo system: she considers a noise on the dynamics of X.

In fact, noise can be introduced in both equations to model different types of randomness: when the noise is on the first equation (dynamics of X) with a standard deviation $\sigma_X > 0$, it models a noisy presynaptic current. When it is on the second equation (dynamics of C) with a standard deviation $\sigma_C > 0$, it describes a noisy conductance dynamic (a noise in the chemical behavior). In general, noise in this model is additive. Various mathematical questions can be studied. Some authors choose to focus on the properties of the natural macroscopic limit of the model as $N \to \infty$ when it is clearly defined (see system (1.2)) while others work on properties of the particle system for fixed N. These models can be quite complicated to study mathematically. The main objectives are to characterize the behavior of these models when the number of neurons N tends to $+\infty$ in a mean-field limit, and to prove whether or not there exists an equilibrium, a stationary behavior, when t tends to $+\infty$. The question of the synchronization of neurons can also be studied, since it is a phenomenon observed in different contexts, such as the generation of respiratory rhythm or complex neurological functionalities. It can be characterized as the dissipation of the empirical variance of the system of neurons. We refer the reader to [6] for further discussion on the synchronization in neuron models, and especially in the Hodgkin-Huxley model.

In [31], the authors work on the determination of firing times. They consider a stochastic FitzHugh-Nagumo model for one neuron, with Brownian noise on X, obtain an approximation of firing times, and compare them with numerical simulations.

In [29], Tatchim Bemmo, Siewe Siewe, and Tchawoua focus on a quite different stochastic model by considering additive noise η on the dynamics of X, and multiplica-

tive noise ξ on the dynamics of C, both defined as sinusoidal functions of correlated Brownian motions. They choose to avoid Gaussian noises since they are unbounded. They also consider a deterministic and periodic entrance signal in the first equation, and observe abrupt transitions of the membrane potential X when the intensity of the noise is gradually changed.

In general, a lot of authors focus on noise on only one variable. In [15], León and Samson consider a FitzHugh-Nagumo model with noise on C but not on X, i.e. $\sigma_X=0$, and study the properties of the equations for one neuron. In particular, they focus on the hypoellipticity of the model, the existence and uniqueness of an invariant probability, and a mixing property by establishing a link between the model and the class of stochastic damping Hamiltonian systems. They also consider neuronal modeling questions and study the generation of spikes according to the parameters of the model. On the contrary, the article [32] focuses on the stochastic FitzHugh-Nagumo model with noise in the dynamics of X, and $\sigma_C=0$. They study one neuron in a periodically forced regime. This study relies on the theory of Markovian Random Dynamical Systems. The model is driven by a cosine signal, and Uda studies the spike rate and compares it with the probability of a two-point motion of membrane potential.

However, some do study stochastic models with two noises. Berglund and Landon describe the behavior of the deterministic FitzHugh-Nagumo model for one neuron in [3], and consider the stochastic model, with noise on both equations, to work on the behavior of the interspike interval and the distribution of oscillations of the solution.

As said above, we consider mean-field interactions. These interactions are described by two functions K_X and K_C , applied on the difference between two states $((X_t^i, C_t^i) - (X_t^j, C_t^j))$. In particular, this type of interaction models electrical synapses.

In their article [1], Baladron, Fasoli, Faugeras, and Touboul study FitzHugh-Nagumo and Hodgkin-Huxley models with mean-field interaction, only on X. They consider more general interactions, not only applied to the difference between two states, modeling chemical synapses and electrical synapses. For the FitzHugh-Nagumo model, they consider a noise on X and prove propagation of chaos, i.e. the convergence of the law of k neurons towards the law of k independent solutions of the mean-field equations. This article is completed and clarified by the work of Bossy, Faugeras, and Talay in [5]. Mischler, Quininao, and Touboul consider a FitzHugh-Nagumo model in [24], with a linear interaction on X, and a noise only on X, i.e. $\sigma_C = 0$ and $K_X(z) = \lambda x$. The drift on X is not exactly the same as in the model above but remains similar as it is a cubic function of X. They work on the properties of a solution of the McKean-Vlasov PDE associated to this model and obtain the uniqueness of a global weak solution. Furthermore, they prove that there exists at least one stationary solution, and when the interaction is small, the stationary solution is unique and exponentially stable. They also exhibit numerical results with open problems, like attractive periodic solutions in time. In a similar framework, Lucon and Poquet study the macroscopic limit of this mean-field model in [20], and in particular the periodicity of such a system. They analyze the influence of both noise and interaction on the emergence of periodic behavior and prove the existence of a periodic solution, exponentially attractive, when the parameters satisfy some assumptions and the drift is "small" enough with respect to interaction and noise. Their approach relies on a slow-fast analysis and Floquet theory. Results of non-uniform propagation of chaos has also been obtained in [23] by Mehri et al., for stochastic spatially structured neuron networks, by applying the Euler approximation to the construction of a solution.

This model can be complexified, by considering non-mean-field interaction. In particular, Bayrak, Hövel, and Vuksanović work on a stochastic FitzHugh-Nagumo model with a network interaction in [2]. Their type of interaction takes into account a connectivity

coefficient between two neurons and a propagation velocity.

Other authors choose to complexify the model by considering stochastic FitzHugh-Nagumo with a spatial model. A second spatial derivative of X is added to the dynamics of X. Various authors study the behavior of such a model and explore the notion of random attractors [22, 16, 18, 17].

Various authors also study numerical schemes for the interacting particle system in the stochastic model. In [25], the authors adapt the Euler-Maruyama scheme to approximate the solution of the particle system.

1.2 Framework and results

Combining noise and interaction, we work specifically on the following equations, for $1 \le i \le N$, where N is the number of neurons

$$\begin{cases} dX_t^{i,N} = (X_t^{i,N} - (X_t^{i,N})^3 - C_t^{i,N} - \alpha)dt + \frac{1}{N} \sum_{j=1}^N K_X(Z_t^{i,N} - Z_t^{j,N})dt + \sigma_X dB_t^{i,X} \\ dC_t^{i,N} = (\gamma X_t^{i,N} - C_t^{i,N} + \beta)dt + \frac{1}{N} \sum_{j=1}^N K_C(Z_t^{i,N} - Z_t^{j,N})dt + \sigma_C dB_t^{i,C}, \end{cases}$$

$$(1.1)$$

where we denote by $Z^{i,N}_t$ the couple $(X^{i,N}_t,C^{i,N}_t)$ to simplify the notation. We assume $(B^{i,X}_t)_i$ and $(B^{i,C}_t)_i$ to be independent Brownian motions. Here, we consider two Brownian noises B^X and B^C , one on each equation, and thus assume that each neuron has its own independent noise and that there is no environmental (or shared) noise.

We also assume K_X and K_C to be Lipschitz continuous and respectively denote their Lipschitz constants by L_X and L_C .

The goal of this article is to describe the behavior of this network as the number Nof neurons tends to infinity.

To describe its behavior, we consider the \mathbb{R}^2 -valued process $(\bar{Z}_t)_{t>0}=(\bar{X}_t,\bar{C}_t)_{t>0}$ evolving according to the following non-linear stochastic differential equation of McKean-Vlasov type

$$\begin{cases} d\bar{X}_{t} = (\bar{X}_{t} - (\bar{X}_{t})^{3} - \bar{C}_{t} - \alpha)dt + K_{X} * \bar{\mu}_{t}(\bar{Z}_{t})dt + \sigma_{X}d\bar{B}_{t}^{X} \\ d\bar{C}_{t} = (\gamma\bar{X}_{t} - \bar{C}_{t} + \beta)dt + K_{C} * \bar{\mu}_{t}(\bar{Z}_{t})dt + \sigma_{C}d\bar{B}_{t}^{C}, \end{cases}$$
(1.2)

where $\bar{\mu}_t = \text{Law}(\bar{Z}_t)$ is the law at time t of the process (\bar{X}_t, \bar{C}_t) , and * denotes the operation of convolution, i.e.

$$K_X * \bar{\mu}_t(u) = \int K_X(u - v)\bar{\mu}_t(dv).$$

To some extent, (1.1) can be seen as an approximation of (1.2) in which the operation of convolution is applied to the empirical measure $\mu_{t, \text{emp}} = \frac{1}{N} \sum_{i=1}^N \delta_{Z_t^{i,N}}$, and what we wish to prove is that, indeed, the law μ_t^N of the network (1.1) converges in some sense to $\bar{\mu}_t^{\otimes N}$ (i.e the law of the solution of (1.2) tensorized N times) as N tends to infinity. This phenomenon has been stated under the name propagation of chaos -an idea motivated by M. Kac [13]- as it amounts to saying that, as the number of particles increases in the system, two particles will become "more and more" independent, their joint law converging towards a tensorized law. The notion of "propagation" refers to the fact that proving such convergence at time 0 is sufficient to prove it at a later time t.

In order to prove the convergence of μ_t^N to $\bar{\mu}_t^{\otimes N}$, we follow the coupling method described in recent work by one of the authors in [10], the result of which cannot be applied directly here. This method has been put forward by Eberle, following earlier works by Lindvall and Rogers [19]. Before recalling the method, let us also mention the recent work [27], which uses a coupling approach adapted to a well-chosen distance.

We consider $r_t^i = |\bar{X}_t^i - X_t^{i,N}| + \delta |\bar{C}_t^i - C_t^{i,N}|$ with $\delta > 0$, a constant not yet specified (to prove the first result Theorem 1.3, we will consider $\delta = 1$, but we will need a more specific one for Theorem 1.4).

A natural distance between probability measures is the Wasserstein distance, linked to the theory of optimal transport. For μ and ν two probability measures on \mathbb{R}^d , we denote

$$W_p(\mu, \nu) = \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}(||X - Y||_p^p)^{1/p},$$
(1.3)

where $||\cdot||_p$ denotes the usual L^p distance on \mathbb{R}^d . It is thus defined as the minimum over all possible choices of a pair (X,Y), such that X is distributed according to μ and Y according to ν , of the expectation of the distance between X and Y. The basic idea behind a coupling method is then that an upper bound on the Wasserstein distance between μ and ν is given by the construction of any pair of random variables distributed according to these probability measures. Thus, instead of considering the minimum over all possible couplings, we construct simultaneously two solutions of (1.1) and (1.2) that will tend to get closer together as the number of neurons increases.

Let $(\bar{X}^i_t, \bar{C}^i_t)$, for i between 1 and N, be N independent copies of a solution of (1.2) driven by some independent Brownian motions $(\bar{B}^{i,X}_t)_{t\geqslant 0}$ and $(\bar{B}^{i,C}_t)_{t\geqslant 0}$. A coupling of $(\bar{X}^i_t, \bar{C}^i_t)$ and $(X^{i,N}_t, C^{i,N}_t)$ then follows from a coupling of the Brownian motions B and \bar{B} .

The first natural choice, popularized by Sznitman [28], is the synchronous coupling and consists in choosing $B=\bar{B}$. By doing so, when considering the time evolution of $\bar{Z}^i_t-Z^{i,N}_t=\left(\bar{X}^i_t-X^{i,N}_t,\bar{C}^i_t-C^{i,N}_t\right)$, the noise cancels out. The contraction of a distance between the processes can then only be induced by the deterministic drift, as in [4], and this usually only holds under rather restrictive conditions (in particular the drift should be strongly convex). Nevertheless, in our case, the calculation of the evolution of $\bar{X}^i_t-X^{i,N}_t$ and $\bar{C}^i_t-C^{i,N}_t$ (see later) shows that there is still some deterministic contraction when $\bar{X}^i_t-X^{i,N}_t=0$. We can therefore use a synchronous coupling in the vicinity of this subspace.

Outside of this subspace, we use the noise to get the processes closer together. In the direction orthogonal to the contracting space we consider $B=-\bar{B}$, as this maximizes the variance of the noise. This yields the *reflection* coupling. Notice however at this stage that, because of the symmetry of the noise, there is a *priori* no reason why r_t^i should decrease rather than increase. This invites us to consider $f(r_t^i)$, with f a concave function, so that a random decrease has more effect than a random increase of the same value. We will define the function f later.

Finally, we construct a Lyapunov function H to take into account the trend of each process to come back to some compact set of \mathbb{R}^2 . We are then led to the study of a suitable distance between the two processes, which will be of the form $\rho_t := \frac{1}{N} \sum_{i=1}^N f(r_t^i)(1+\epsilon H(\bar{Z}_t^i)+\epsilon H(Z_t^{i,N}))$, where $\epsilon>0$. This quantity controls the usual L^1 and L^2 distances between the two systems and is interesting as, when r_t^i is small, $f(r_t^i)$ tends to decrease either because of the deterministic drift or the reflection coupling, and when r_t^i is big, the Lyapunov functions H will tend to decrease. We thus show that $\mathbb{E}\rho_t$ decays exponentially fast. This leads to several constraints on δ, ϵ and on the parameters involved in the definition of f, and we have to prove that it is possible to meet all these conditions simultaneously. In reality, the quantity ρ_t considered will be a slight twist of the one given above (see (2.25)) so as to take into account the non-linearity of the process.

As explained, this method requires some noise in the direction orthogonal to the naturally contracting subspace. This means, in the description of the method above, that one should have $\sigma_X>0$ (so that we can use a reflection coupling to bring \bar{X}^i_t and

 $X_t^{i,N}$ closer together). In the case $\sigma_X=0$ and $\sigma_C>0$, a modification of the calculations is necessary. We describe this case and the resulting modifications in the computations in Appendix B.

Assumption 1.1. K_X and K_C are Lipschitz continuous, i.e.

$$\exists L_X \ge 0, \forall z, z' \in \mathbb{R}^2 \quad |K_X(z) - K_X(z')| \le L_X(\|z - z'\|_1)$$

$$\exists L_C \ge 0, \forall z, z' \in \mathbb{R}^2 \quad |K_C(z) - K_C(z')| \le L_C(\|z - z'\|_1)$$

$$K_X(0, 0) = 0 \text{ and } K_C(0, 0) = 0.$$

Before any result on the propagation of chaos, we prove that both systems (1.1) and (1.2) have well-defined solutions.

Proposition 1.2 (Existence of solutions). Let K_X and K_C satisfy Assumption 1.1. We assume the law of $\left((X_0^{1,N},C_0^{1,N}),\ldots,(X_0^{N,N},C_0^{N,N})\right)$ and the law of (\bar{X}_0,\bar{C}_0) have a moment of order 2. Then, there exists a unique strong solution for system (1.1) and a unique strong solution for system (1.2).

We denote W_1 and W_2 the usual L^1 and L^2 Wasserstein distances defined in (1.3).

Theorem 1.3 (Non uniform in time propagation of chaos). Let K_X and K_C satisfy Assumption 1.1. There exist explicit $C_1, C_2 > 0$, such that for all probability measures μ_0 on \mathbb{R}^2 with finite second moment,

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \le C_1 e^{C_2 t} \frac{k}{\sqrt{N}},$$

for all $k \in \mathbb{N}$, where $\mu_t^{k,N}$ is the marginal distribution at time t of the first k neurons $\left((X_t^{1,N},C_t^{1,N}),\ldots,(X_t^{k,N},C_t^{k,N})\right)$ of an N-particle system (1.1) with initial distribution $\mu_0^{\otimes N}$, while $\bar{\mu}_t$ is a solution of (1.2) with initial distribution μ_0 .

This first theorem is in accordance with the theorem from [14] and makes the dependence in t explicit. Since its proof is rather quick and provides a good entry point into coupling methods, we give it in Subsection 1.4.

Our main result consists in removing the time dependency in the previous upper bound. This uniform in time propagation of chaos however requires stronger assumptions on the interaction kernels.

Theorem 1.4 (Uniform in time propagation of chaos). Let $L_{X,\max}$ and $L_{C,\max}$ be two (explicit) universal constants such that $L_X \leq L_{X,\max}$ and $L_C \leq L_{C,\max}$. Let $\mathcal{C}_{init,exp} > 0$ and $\tilde{a} > 0$. There is an explicit $c^K > 0$ such that, for all K_X and K_C satisfying Assumptions 1.1 with $L_X, L_C < c^K$, there exist explicit $B_1, B_2 > 0$, such that for all probability measures μ_0 on \mathbb{R}^2 satisfying $\mathbb{E}_{\mu_0}\left(e^{\tilde{a}(|X|+|C|)}\right) \leq \mathcal{C}_{init,exp}$,

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq B_1 \frac{k}{\sqrt{N}}, \qquad \mathcal{W}_2^2\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq B_2 \frac{k}{\sqrt{N}},$$

for all $k \in \mathbb{N}$, where $\mu_t^{k,N}$ is the marginal distribution at time t of the first k neurons $\left((X_t^{1,N},C_t^{1,N}),\ldots,(X_t^{k,N},C_t^{k,N})\right)$ of an N-particle system (1.1) with initial distribution $\mu_0^{\otimes N}$, while $\bar{\mu}_t$ is a solution of (1.2) with initial distribution μ_0 .

When we prove uniform in time propagation of chaos, $L_{X,\max}$ and $L_{C,\max}$ are a priori bounds. Theorem 1.4 above will be true for L_X and L_C sufficiently small: the condition $L_X \leq L_{X,\max}$ and $L_C \leq L_{C,\max}$ are therefore not restrictive conditions and are useful in proving some parameters are independent of L_X and L_C . Lemma 2.2 below shows that one can for instance consider $L_{X,\max} = 4$ and $L_{C,\max} = \frac{1}{5}$. Furthermore c^K , that controls both interactions K_X and K_C , is explained in Subsection 2.4.

The main interest of obtaining uniform in time estimates is that it allows the study and comparison of the long-time behavior of the particle system and its nonlinear limit. As previously mentioned, this work follows the method described in [10]. Beyond the result of uniform in time propagation of chaos for the FitzHugh-Nagumo model, which is in itself an interesting result, the present work is also a testimony to the robustness of the coupling method.

The reader will find **an index containing the notation**, constants, and parameters for reference at the end of the document.

1.3 Existence of solutions

First of all, we prove Proposition 1.2, i.e existence of strong solutions of systems (1.1) and (1.2), under Assumption 1.1. Let's denote, for $\kappa \in \mathbb{R}^+$,

$$g_{\kappa}(x) = \begin{cases} -\kappa^3 & \text{if } x < -\kappa \\ x^3 & \text{if } x \in [-\kappa, \kappa] \\ \kappa^3 & \text{if } x > \kappa. \end{cases}$$

 g_{κ} is Lipschitz and is bounded.

Thus, it's well known (see Chapter 4 [12]) that the following system (under Assumption 1.1 and the assumption that the initial condition has a moment of order 2)

on 1.1 and the assumption that the initial condition has a moment of order 2)
$$\begin{cases} dX_t^{i,N,\kappa} = & (X_t^{i,N,\kappa} - g_{\kappa}(X_t^{i,N,\kappa}) - C_t^{i,N,\kappa} - \alpha)dt + \frac{1}{N} \sum_{j=1}^N K_X(Z_t^{i,N,\kappa} - Z_t^{j,N,\kappa}) \\ + \sigma_X dB_t^{i,X} \\ dC_t^{i,N,\kappa} = & (\gamma X_t^{i,N,\kappa} - C_t^{i,N,\kappa} + \beta)dt + \frac{1}{N} \sum_{j=1}^N K_C(Z_t^{i,N,\kappa} - Z_t^{j,N,\kappa}) + \sigma_C dB_t^{i,C}, \end{cases}$$

$$(1.4)$$

for $1 \leq i \leq N$, has a strong and unique solution that we denote $(X_t^{i,N,\kappa},C_t^{i,N,\kappa})_{1\leq i\leq N}$. In consequence, for a fixed $\kappa\in\mathbb{R}^+$, there exists a strong solution of system (1.1) until time

$$T_{\kappa} = \sup \left\{ t, \forall i, \forall s \leq t, X_s^{i,N,\kappa} \leq \kappa \text{ and } C_s^{i,N,\kappa} \leq \kappa \right\},$$

and the solution coincides with the solution of the system with g_{κ} .

We have the following Lemma

Lemma 1.5. If, for each $i \leq N$, $\mathbb{E}(|X_0^{i,N,\kappa}|^2) < +\infty$ and $\mathbb{E}(|C_0^{i,N,\kappa}|^2) < +\infty$, then for all $t \geq 0$ there exists $C_t < \infty$ such that, for each $i \leq N$

$$\mathbb{E}\left(|X_t^{i,N,\kappa}|^2 + |C_t^{i,N,\kappa}|^2\right) \le C_t. \tag{1.5}$$

The proof relies on the Lyapunov function defined in the next Section and is given in Appendix A.3.

Then, by denoting T_{∞} the explosion time of a solution of system (1.1)

$$T_{\infty} = \inf \left\{ t, \exists i, \forall A > 0, X_t^{i,N,\kappa} > A \text{ or } C_t^{i,N,\kappa} > A \right\}$$

we obtain that $\forall t \in \mathbb{R}^+, \mathbb{P}(T_\infty \leq t) = 0$ and $\mathbb{P}(\bar{T}_\infty \leq t) = 0$. Eventually, there exists a unique and strong solution of system (1.1).

The existence and uniqueness of a solution of (1.2) is known from the Theorem 3.3 from [7], under the assumption that the law of the initial point (\bar{X}_0,\bar{C}_0) has a moment of order 2. We only have to prove that Assumptions 3.2 [7] are satisfied. We define, for all $t \in \mathbb{R}^+$, $z = (x,c) \in \mathbb{R}^2$ and for all probability distribution ν with a finite variance

$$b(t,z,\nu) = \begin{pmatrix} x-x^3-c-\alpha+K_X*\nu(z)\\ \gamma x-c+\beta+K_C*\nu(z) \end{pmatrix} \quad \text{ and } \quad \sigma(t,z,\nu) = \begin{pmatrix} \sigma_X\\ \sigma_C \end{pmatrix}.$$

 σ is a constant function, so it clearly satisfies the various conditions. For $t \in \mathbb{R}^+$, z, z' in \mathbb{R}^2 , and ν a probability measure

$$\langle z - z', b(t, z, \nu) - b(t, z', \nu) \rangle$$

$$= (x - x') \left[(x - x') - (x^3 - x'^3) - (c - c') + K_X * \nu(z) - K_X * \nu(z') \right]$$

$$+ (c - c') \left(\gamma(x - x') - (c - c') + K_C * \nu(z) - K_C * \nu(z') \right)$$

$$= (x - x')^2 - (x - x')^2 (x^2 + xx' + x'^2) + (\gamma - 1)(c - c')(x - x') - (c - c')^2$$

$$+ (x - x')(K_X * \nu(z) - K_X * \nu(z')) + (c - c')(K_C * \nu(z) - K_C * \nu(z')).$$

Since $x^2+xx'+x'^2\geq 0$, the second term is non-positive. K_X and K_C are Lipschitz continuous functions, so the last line is clearly bounded by $\|z-z'\|_2^2$ up to a multiplicative constant. Then, there exists a constant L such that

$$\langle z - z', b(t, z, \nu) - b(t, z', \nu) \rangle \le L \|z - z'\|_2^2$$

Since K_X and K_C are Lipschitz continuous functions, we also have, for all probability distributions ν and ν' with a finite variance,

$$||b(t, z, \nu) - b(t, z, \nu')||_2 \le LW_2(\nu, \nu').$$

Eventually, since b is locally Lipschitz continuous with polynomial growth, each Assumption is satisfied and Theorem 3.3 [7] can be applied. Note that we could also apply Proposition 2.19 from [21]: assumptions are the same, and it gives a result for interaction depending on a spatial position.

To complete the Lemma 1.5, we also give the following

Proposition 1.6. If, for each $i \leq N$, $\mathbb{E}(|X_0^{i,N}|^2) < +\infty$ and $\mathbb{E}(|C_0^{i,N}|^2) < +\infty$, then for all $t \geq 0$ there exists $\mathcal{C}_t < \infty$ such that, for each $i \leq N$

$$\mathbb{E}\left(|X_t^{i,N}|^2 + |C_t^{i,N}|^2\right) \le C_t. \tag{1.6}$$

and

Proposition 1.7. If $\mathbb{E}(|\bar{X}_0|^2) < +\infty$ and $\mathbb{E}(|\bar{C}_0|^2) < +\infty$, then there exist $C_{0,1}$ and $C_{0,2}$ such that

$$\mathbb{E}\left(|\bar{X}_t|^2 + |\bar{C}_t|^2\right) \le C_{0,1}e^{C_{0,2}t}.\tag{1.7}$$

The proof is very similar to the proof of Lemma 1.5 and can be found in Appendix A.3.

1.4 Quick result: non uniform in time propagation of chaos

We start by proving Theorem 1.3, a non uniform in time propagation of chaos, as it highlights the basic strategy behind a coupling argument. Some of the following expressions will be used in the proof of Theorem 1.4, in Section 3.

We consider a synchronous coupling between $(Z_t^{i,N})_i$ and $(\bar{Z}_t^i)_i$, i.e. for each $1 \leq i \leq N$, we choose $\tilde{B}_t^{i,X} = B_t^{i,X}$ and $\tilde{B}_t^{i,\bar{C}} = B_t^{i,C}$. We have

$$\begin{cases} dX_t^{i,N} = \left(X_t^{i,N} - (X_t^{i,N})^3 - C_t^{i,N} - \alpha\right) dt + \frac{1}{N} \sum_{j=1}^N K_X(Z_t^{i,N} - Z_t^{j,N}) dt + \sigma_X dB_t^{i,X} \\ dC_t^{i,N} = \left(\gamma X_t^{i,N} - C_t^{i,N} + \beta\right) dt + \frac{1}{N} \sum_{j=1}^N K_C(Z_t^{i,N} - Z_t^{j,N}) dt + \sigma_C dB_t^{i,C} \end{cases}$$

and

$$\begin{cases} d\bar{X}_{t}^{i} = (\bar{X}_{t}^{i} - (\bar{X}_{t}^{i})^{3} - \bar{C}_{t}^{i} - \alpha)dt + K_{X} * \bar{\mu}_{t}(\bar{Z}_{t}^{i})dt + \sigma_{X}dB_{t}^{i,X} \\ d\bar{C}_{t}^{i} = (\gamma \bar{X}_{t}^{i} - \bar{C}_{t}^{i} + \beta)dt + K_{C} * \bar{\mu}_{t}(\bar{Z}_{t}^{i})dt + \sigma_{C}dB_{t}^{i,C}, \end{cases}$$

with $\bar{\mu}_t$ the law of \bar{Z}_t^1 . The method is the following

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- we compute the time evolution of $\mathbb{E}\left(|X_t^{i,N}-\bar{X}_t^i|+|C_t^{i,N}-\bar{C}_t^i|\right)$ using Ito's formula.
- we control the difference between the drifts $\frac{1}{N}\sum_{j\neq i}K(\bar{Z}^i_t-\bar{Z}^j_t)$ and $K*\bar{\mu}_t(\bar{Z}^i_t)$ using some form of the law of large numbers. This is where the convergence rate \sqrt{N} appears,
- · and we conclude using Gronwall's lemma.

Time evolution: We have,

$$d(X_t^{i,N} - \bar{X}_t^i) = \left((X_t^{i,N} - \bar{X}_t^i) - \left((X_t^{i,N})^3 - (\bar{X}_t^i)^3 \right) - (C_t^{i,N} - \bar{C}_t^i) + \frac{1}{N} \sum_{i=1}^N K_X (Z_t^{i,N} - Z_t^{j,N}) - K_X * \bar{\mu}_t(\bar{Z}_t^i) \right) dt.$$

We denote

$$sign(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and obtain, using Ito's formula for a twice continuously differentiable approximation of the absolute value and usual convergence lemmas (see Lemma A.1 below),

$$d|X_{t}^{i,N} - \bar{X}_{t}^{i}| = \left(\operatorname{sign}(X_{t}^{i,N} - \bar{X}_{t}^{i})(X_{t}^{i,N} - \bar{X}_{t}^{i}) - \operatorname{sign}(X_{t}^{i,N} - \bar{X}_{t}^{i})\left((X_{t}^{i,N})^{3} - (\bar{X}_{t}^{i})^{3}\right) - \operatorname{sign}(X_{t}^{i,N} - \bar{X}_{t}^{i})(C_{t}^{i,N} - \bar{C}_{t}^{i}) + \operatorname{sign}(X_{t}^{i,N} - \bar{X}_{t}^{i})\frac{1}{N}\sum_{j=1}^{N}K_{X}(Z_{t}^{i,N} - Z_{t}^{j,N}) - K_{X} * \bar{\mu}_{t}(\bar{Z}_{t}^{i})\right)dt$$

$$\leq \left(|X_{t}^{i,N} - \bar{X}_{t}^{i}| - \left|(X_{t}^{i,N})^{3} - (\bar{X}_{t}^{i})^{3}\right| + \left|C_{t}^{i,N} - \bar{C}_{t}^{i}\right| + \left|\frac{1}{N}\sum_{j=1}^{N}K_{X}(Z_{t}^{i,N} - Z_{t}^{j,N}) - K_{X} * \bar{\mu}_{t}(\bar{Z}_{t}^{i})\right|\right)dt. \tag{1.8}$$

Similarly

$$d|C_{t}^{i,N} - \bar{C}_{t}^{i}| \leq \left(\gamma \left| X_{t}^{i,N} - \bar{X}_{t}^{i} \right| - |C_{t}^{i,N} - \bar{C}_{t}^{i}| + \left| \frac{1}{N} \sum_{j=1}^{N} K_{C}(Z_{t}^{i,N} - Z_{t}^{j,N}) - K_{C} * \bar{\mu}_{t}(\bar{Z}_{t}^{i}) \right| \right) dt.$$

$$(1.9)$$

Thus, denoting $r^i_t=|X^{i,N}_t-\bar{X}^i_t|+|C^{i,N}_t-\bar{C}^i_t|$ (i.e considering $\delta=1$) we obtain,

$$\begin{split} dr_t^i & \leq \left((1+\gamma) |X_t^{i,N} - \bar{X}_t^i| - \left| \left(X_t^{i,N} \right)^3 - \left(\bar{X}_t^i \right)^3 \right| \\ & + \left| \frac{1}{N} \sum_{j=1}^N K_X (Z_t^{i,N} - Z_t^{j,N}) - K_X * \bar{\mu}_t (\bar{Z}_t^i) \right| \\ & + \left| \frac{1}{N} \sum_{j=1}^N K_C (Z_t^{i,N} - Z_t^{j,N}) - K_C * \bar{\mu}_t (\bar{Z}_t^i) \right| \right) dt. \end{split}$$

Difference of the drifts: Let us now consider these last two terms

$$\left| \frac{1}{N} \sum_{j=1}^{N} K_X(Z_t^{i,N} - Z_t^{j,N}) - K_X * \bar{\mu}_t(\bar{Z}_t^i) \right| \le \left| \frac{1}{N} \sum_{j=1}^{N} K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i) \right|$$

$$+ \left| \frac{1}{N} \sum_{j=1}^{N} K_X(Z_t^{i,N} - Z_t^{j,N}) - \frac{1}{N} \sum_{j=1}^{N} K_X(\bar{Z}_t^i - \bar{Z}_t^j) \right|.$$

The first sum can be decomposed, using Assumption 1.1, into

$$\frac{1}{N} \left| \sum_{j=1}^{N} K_X(Z_t^{i,N} - Z_t^{j,N}) - K_X(\bar{Z}_t^i - \bar{Z}_t^j) \right| \leq \frac{L_X}{N} \sum_{j=1}^{N} \|Z_t^{i,N} - Z_t^{j,N} - (\bar{Z}_t^i - \bar{Z}_t^j)\|_1 \\
\leq L_X r_t^i + \frac{L_X}{N} \sum_{j=1}^{N} r_t^j.$$

Similarly, we obtain

$$\left| \frac{1}{N} \sum_{j=1}^{N} K_C(Z_t^{i,N} - Z_t^{j,N}) - K_C * \bar{\mu}_t(\bar{Z}_t^i) \right| \le L_C r_t^i + \frac{L_C}{N} \sum_{j=1}^{N} r_t^j + \left| \frac{1}{N} \sum_{j=1}^{N} K_C(\bar{Z}_t^i - \bar{Z}_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i) \right|.$$

Hence, we get

$$dr_{t}^{i} \leq \left((1+\gamma)r_{t}^{i} + (L_{X} + L_{C}) \left(r_{t}^{i} + \frac{1}{N} \sum_{j=1}^{N} r_{t}^{j} \right) + \left| \frac{1}{N} \sum_{j=1}^{N} K_{X} (\bar{Z}_{t}^{i} - \bar{Z}_{t}^{j}) - K_{X} * \bar{\mu}_{t}(\bar{Z}_{t}^{i}) \right| + \left| \frac{1}{N} \sum_{j=1}^{N} K_{C} (\bar{Z}_{t}^{i} - \bar{Z}_{t}^{j}) - K_{C} * \bar{\mu}_{t}(\bar{Z}_{t}^{i}) \right| dt.$$

By considering the expectation, since $\mathbb{E}(r_t^j) = \mathbb{E}(r_t^i)$ for each j, by exchangeability of the particles, we have

$$d\mathbb{E}(r_t^i) \le \left((1 + \gamma + 2L_X + 2L_C) \mathbb{E}(r_t^i) + \mathbb{E}\left[\left| \frac{1}{N} \sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i) \right| \right] + \mathbb{E}\left[\left| \frac{1}{N} \sum_{j=1}^N K_C(\bar{Z}_t^i - \bar{Z}_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i) \right| \right] \right) dt.$$

Now, we bound the interaction part. We begin with K_X . By Cauchy-Schwarz

inequality, we can write

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^{N}K_{X}(\bar{Z}_{t}^{i}-\bar{Z}_{t}^{j})-K_{X}*\bar{\mu}_{t}(\bar{Z}_{t}^{i})\right|\right]$$

$$\leq \mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_{X}(\bar{Z}_{t}^{i}-\bar{Z}_{t}^{j})-K_{X}*\bar{\mu}_{t}(\bar{Z}_{t}^{i})\right|^{2}\right)^{1/2}$$

We notice that $(\bar{Z}_t^j)_j$ are i.i.d with law $\bar{\mu}_t$. Let's denote \bar{Z}_t a generic random variable of law $\bar{\mu}_t$ independent of \bar{Z}_t^i . What is more, $K_X*\bar{\mu}_t(\bar{Z}_t^i)=\int K_X(\bar{Z}_t^i-z)\bar{\mu}_t(dz)=\mathbb{E}[K_X(\bar{Z}_t^i-\bar{Z}_t)|\bar{Z}_t^i]$. Hence

$$\begin{split} \mathbb{E}\left(\mathbb{E}\left(\left|\frac{1}{N-1}\sum_{j\neq i}K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|^2\left|\bar{Z}_t^i\right)\right)\\ &=\mathbb{E}\left(\operatorname{Var}\left(\frac{1}{N-1}\sum_{j\neq i}K_X(\bar{Z}_t^i-\bar{Z}_t^j)\middle|\bar{Z}_t^i\right)\right)\\ &\leq \frac{L_X^2}{N-1}\mathbb{E}\left(\operatorname{Var}\left(\|\bar{Z}_t^i-\bar{Z}_t\|_1\middle|\bar{Z}_t^i\right)\right). \end{split}$$

Since

$$\begin{split} \mathbb{E}\left[\operatorname{Var}\left(\|\bar{Z}_t^i - \bar{Z}_t\|_1 \Big| \bar{Z}_t^i \right) \right] \leq & \mathbb{E}\left[\mathbb{E}\left(\|\bar{Z}_t^i - \bar{Z}_t\|_1^2 \Big| \bar{Z}_t^i \right) \right] \\ \leq & \mathbb{E}\left[\mathbb{E}\left(2 \|\bar{Z}_t^i\|_1^2 + 2 \|\bar{Z}_t\|_1^2 \Big| \bar{Z}_t^i \right) \right] \leq 4 \mathbb{E}(\|\bar{Z}_t\|_1^2), \end{split}$$

we obtain

$$\mathbb{E}\left(\mathbb{E}\left(\left|\frac{1}{N-1}\sum_{j\neq i}K_{X}(\bar{Z}_{t}^{i}-\bar{Z}_{t}^{j})-K_{X}*\bar{\mu}_{t}(\bar{Z}_{t}^{i})\right|^{2}\left|\bar{Z}_{t}^{i}\right)\right)\leq\frac{4L_{X}^{2}}{N-1}\mathbb{E}(\|\bar{Z}_{t}\|_{1}^{2}).$$

We now want to control $\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|^2\right)$. We decompose it into

$$\begin{split} &\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_{X}(\bar{Z}_{t}^{i}-\bar{Z}_{t}^{j})-K_{X}*\bar{\mu}_{t}(\bar{Z}_{t}^{i})\right|^{2}\right)\\ =&\mathbb{E}\left(\left|\frac{N-1}{N}\frac{1}{N-1}\sum_{j=1}^{N}K_{X}(\bar{Z}_{t}^{i}-\bar{Z}_{t}^{j})-\left(\frac{N-1}{N}+\frac{1}{N}\right)K_{X}*\bar{\mu}_{t}(\bar{Z}_{t}^{i})\right|^{2}\right)\\ \leq&2\frac{(N-1)^{2}}{N^{2}}\mathbb{E}\left(\left|\frac{1}{N-1}\sum_{j=1}^{N}K_{X}(\bar{Z}_{t}^{i}-\bar{Z}_{t}^{j})-K_{X}*\bar{\mu}_{t}(\bar{Z}_{t}^{i})\right|^{2}\right)\\ &+\frac{2}{N^{2}}\mathbb{E}\left(|K_{X}*\bar{\mu}_{t}(\bar{Z}_{t}^{i})|^{2}\right). \end{split}$$

Since

$$\mathbb{E}\left(|K_X * \bar{\mu}_t(\bar{Z}_t^i)|^2\right) = \mathbb{E}\left(\left|\mathbb{E}\left(K_X(\bar{Z}_t^i - \bar{Z}_t)|\bar{Z}_t^i\right)\right|^2\right)$$

$$\leq L_X^2 \mathbb{E}\left(\mathbb{E}\left(\|\bar{Z}_t^i - \bar{Z}_t\|_1^2|\bar{Z}_t^i\right)\right) \leq 4L_X^2 \mathbb{E}(\|\bar{Z}_t\|_1^2),$$

we obtain

$$\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_{X}(\bar{Z}_{t}^{i}-\bar{Z}_{t}^{j})-K_{X}*\bar{\mu}_{t}(\bar{Z}_{t}^{i})\right|^{2}\right) \\
\leq \left(\frac{N-1}{N}\right)^{2}\frac{4L_{X}^{2}}{N-1}\mathbb{E}(\|\bar{Z}_{t}\|_{1}^{2})+\frac{4L_{X}^{2}}{N^{2}}\mathbb{E}(\|\bar{Z}_{t}\|_{1}^{2})\leq \frac{8L_{X}^{2}}{N}\mathbb{E}(\|\bar{Z}_{t}\|_{1}^{2}), \tag{1.10}$$

and finally

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^{N}K_{X}(\bar{Z}_{t}^{i}-\bar{Z}_{t}^{j})-K_{X}*\bar{\mu}_{t}(\bar{Z}_{t}^{i})\right|\right]\leq \left(\frac{8L_{X}^{2}}{N}\mathbb{E}(\|\bar{Z}_{t}\|_{1}^{2})\right)^{1/2}.$$

Similarly, we have

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^{N}K_{C}(\bar{Z}_{t}^{i}-\bar{Z}_{t}^{j})-K_{C}*\bar{\mu}_{t}(\bar{Z}_{t}^{i})\right|\right] \leq \left(\frac{8L_{C}^{2}}{N}\mathbb{E}(\|\bar{Z}_{t}\|_{1}^{2})\right)^{1/2},$$

which yields

$$d\mathbb{E}(r_t^i) \le \left((1 + \gamma + 2L_X + 2L_C) \mathbb{E}(r_t^i) + \sqrt{8L_X^2 + 8L_C^2} \left(\frac{1}{N} \mathbb{E}(\|\bar{Z}_t\|_1^2) \right)^{1/2} \right) dt.$$

Then using Proposition 1.7, we obtain

$$d\mathbb{E}(r_t^i) \le \left((1 + \gamma + 2L_X + 2L_C) \mathbb{E}(r_t^i) + \frac{\sqrt{8L_X^2 + 8L_C^2} \sqrt{2C_{0,1}}}{\sqrt{N}} e^{\frac{1}{2}C_{0,2}t} \right) dt$$

Conclusion: We have thus obtained

$$\begin{split} d\left(\mathbb{E}(r_t^i) + \sqrt{\frac{16(L_X^2 + L_C^2)C_{0,1}}{N}} \frac{1}{1 + \gamma + 2L_X + 2L_C - \frac{C_{0,2}}{2}} e^{\frac{1}{2}C_{0,2}t}\right) \\ & \leq (1 + \gamma + 2L_X + 2L_C) \\ & \times \left(\mathbb{E}(r_t^i) + \sqrt{\frac{16(L_X^2 + L_C^2)C_{0,1}}{N}} \frac{1}{1 + \gamma + 2L_X + 2L_C - \frac{C_{0,2}}{2}} e^{\frac{1}{2}C_{0,2}t}\right) dt, \end{split}$$

and Gronwall's lemma yields

$$\begin{split} \mathbb{E}(r_t^i) + \sqrt{\frac{16(L_X^2 + L_C^2)C_{0,1}}{N}} \frac{1}{1 + \gamma + 2L_X + 2L_C - \frac{C_{0,2}}{2}} e^{\frac{1}{2}C_{0,2}t} \\ \leq & e^{(1 + \gamma + 2L_X + 2L_C)t} \\ & \times \left[\mathbb{E}(r_0^i) + \sqrt{\frac{16(L_X^2 + L_C^2)C_{0,1}}{N}} \frac{1}{1 + \gamma + 2L_X + 2L_C - \frac{C_{0,2}}{2}} \right], \end{split}$$

thus

$$\mathbb{E}(r_t^i) \le C_1 e^{C_2 t} \frac{1}{\sqrt{N}}.$$

Let μ_0 a measure on \mathbb{R}^2 , $\mu_t^{k,N}$ the marginal distribution at time t of the first k neurons $\left(Z_t^{1,N},\ldots,Z_t^{k,N}\right)$ of an N-particle system (1.1) with initial distribution $\mu_0^{\otimes N}$, and $\bar{\mu}_t$ is a solution of (1.2) with initial distribution μ_0 . We obtain for the L^1 Wasserstein distance

$$\begin{split} \mathcal{W}_{1}(\mu_{t}^{k,N}, \bar{\mu}_{t}^{\otimes k}) &= \inf \left\{ \mathbb{E}[\|Z^{(k)} - \bar{Z}^{(k)}\|_{1}], \mathbb{P}_{Z^{(k)}} = \mu_{t}^{k,N}, \mathbb{P}_{\bar{Z}^{(k)}} = \bar{\mu}_{t}^{\otimes k} \right\} \\ &\leq \inf \left\{ \mathbb{E}\left[\sum_{i=1}^{k} r_{t}^{i}\right], \mathbb{P}_{(Z_{t}^{i,N})_{i}} = \mu_{t}^{k,N}, \mathbb{P}_{(\bar{Z}_{t}^{i})_{i}} = \bar{\mu}_{t}^{\otimes k} \right\} \\ &\leq k \mathbb{E}(r_{t}^{1}) \\ &\leq C_{1} e^{C_{2}t} \frac{k}{\sqrt{N}}. \end{split}$$

We hence obtain Theorem 1.3.

2 Preliminaries

In this section, before tackling the proof by the coupling method of the uniform in time propagation of chaos, we gather the various technical lemmas and construct the necessary objects.

2.1 Notation

To construct the Lyapunov functions (which allow us to bound the moments of the processes and show that they tend to come back to some compact set), we begin by introducing the generators of the processes.

For $h: \mathbb{R}^{2N} \to \mathbb{R}$, for all $(z_i)_{1 \le i \le N} = (x_i, c_i)_{1 \le i \le N} \in \mathbb{R}^{2N}$, the generator of (1.1) is

$$\mathcal{L}^N h(z_1,\ldots,z_N) = \sum_{i=1}^N \mathcal{L}^{i,N} h,$$

where

$$\mathcal{L}^{i,N}h(z_1,\dots,z_N) = \left(x_i - x_i^3 - c_i - \alpha + \frac{1}{N} \sum_{j=1}^N K_X(z_i - z_j)\right) \partial_{x_i}h$$

$$+ \left(\gamma x_i - c_i + \beta + \frac{1}{N} \sum_{j=1}^N K_C(z_i - z_j)\right) \partial_{c_i}h$$

$$+ \frac{\sigma_X^2}{2} \partial_{x_i,x_i}^2 h + \frac{\sigma_C^2}{2} \partial_{c_i,c_i}^2 h.$$

For $h: \mathbb{R}^2 \to \mathbb{R}$, for all $z=(x,y) \in \mathbb{R}^2$, the generator of (1.2) for a given distribution μ is

$$\mathcal{L}_{\mu}h(x,c) = \left(x - x^3 - c - \alpha + K_X * \mu(z)\right) \partial_x h + \left(\gamma x - c + \beta + K_C * \mu(z)\right) \partial_c h + \frac{\sigma_X^2}{2} \partial_{xx}^2 h + \frac{\sigma_C^2}{2} \partial_{cc}^2 h.$$

In particular, we notice that for fixed $(z_i)_{1 \leq i \leq N} \in (\mathbb{R}^2)^N$, if we consider the empirical measure $\{\mu_{\text{emp}} = \frac{1}{N} \sum_j \delta_{z_j} \}$, we have for all $h : \mathbb{R}^2 \to \mathbb{R}$ and $\bar{z} \in \mathbb{R}^2$,

$$\begin{split} \mathcal{L}_{\mu_{\text{emp}}}h(\bar{z}) &= \left(\bar{x} - \bar{x}^3 - \bar{c} - \alpha + K_X * \mu_{\text{emp}}(\bar{z})\right) \partial_x h + \left(\gamma \bar{x} - \bar{c} + \beta + K_C * \mu_{\text{emp}}(\bar{z})\right) \partial_c h \\ &+ \frac{\sigma_X^2}{2} \partial_{xx}^2 h + \frac{\sigma_C^2}{2} \partial_{cc}^2 h \\ &= \left(\bar{x} - \bar{x}^3 - \bar{c} - \alpha + \frac{1}{N} \sum_{j=1}^N K_X(\bar{z} - z_j)\right) \partial_x h \\ &+ \left(\gamma \bar{x} - \bar{c} + \beta + \frac{1}{N} \sum_{j=1}^N K_C(\bar{z} - z_j)\right) \partial_c h + \frac{\sigma_X^2}{2} \partial_{xx}^2 h + \frac{\sigma_C^2}{2} \partial_{cc}^2 h. \end{split}$$

In this case, if we consider $\bar{z}=z_i$ for a specific i and we denote $\bar{h}^i:(z_1,\ldots,z_N)\to h(z_i)$, then

$$\begin{split} \mathcal{L}_{\mu_{\text{emp}}}h(z_i) &= \left(x_i - x_i^3 - c_i - \alpha + \frac{1}{N}\sum_{j=1}^N K_X(z_i - z_j)\right)\partial_x h \\ &+ \left(\gamma x_i - c_i + \beta + \frac{1}{N}\sum_{j=1}^N K_C(z_i - z_j)\right)\partial_c h + \frac{\sigma_X^2}{2}\partial_{xx}^2 h + \frac{\sigma_C^2}{2}\partial_{cc}^2 h \\ &= \mathcal{L}^{i,N}\bar{h}^i(z_1, \dots, z_N). \end{split}$$

2.2 First Lyapunov function

Let $H: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$H(z) = H(x,c) = \frac{1}{2}\gamma x^2 + \beta x + \frac{1}{2}c^2 + \alpha c + H_0,$$
 (2.1)

with

$$H_0 = \frac{\beta^2}{\gamma} + \alpha^2,$$

where γ , β and α are the parameters of the system (1.1).

Lemma 2.1. (i) For all $x, c \in \mathbb{R}$, we have $H(x, c) \ge \frac{\gamma}{4}x^2 + \frac{c^2}{4} \ge 0$,

(ii) For all
$$x, c \in \mathbb{R}$$
, we have $H(x, c) \ge \frac{1}{2 \max(\gamma, 1)} \left((\gamma x + \beta)^2 + (c + \alpha)^2 \right)$,

(iii) For all $\delta>0$ there is $\mathcal{C}_{r,H}>0$ such that for all $x,x',c,c'\in\mathbb{R}$, we have

$$(|x - x'| + \delta |c - c'|)^2 \le C_{r,H}(H(x,c) + H(x',c')),$$

(iv) A direct consequence of the previous point is that for all $B \in \mathbb{R}$, $\lambda > 0$ and $\delta > 0$, there is $R \geq 0$ such that, for $x, x', c, c' \in \mathbb{R}$ satisfying $|x - x'| + \delta |c - c'| \geq R$, we have $H(x,c) + H(x',c') \geq \frac{80B}{\lambda}$. An explicit value of R is given by $R = \sqrt{\frac{1280(1+\delta^2)B}{\lambda \min(\gamma,1)}}$.

The first two points are consequences of direct calculations. The last two points are proved in Appendix A.2. The constant $C_{r,H}$ has been thus named because it ensures the control of the modified Euclidean distance r, precisely defined in (2.24), by the function H.

Lemma 2.2 (Lyapunov's property of H). Let $\lambda \in \mathbb{R}$ such that

$$\frac{L_X}{8} + L_C \left(2 + \frac{1}{8} \right) < 1 - \frac{\lambda}{2},\tag{2.2}$$

then, for H defined in (2.1), there exists B>0 such that for all $(\bar{x},\bar{c})\in\mathbb{R}^2$, for all probability distribution μ on \mathbb{R}^2 ,

$$\mathcal{L}_{\mu}H(\bar{z}) \leq B + (\alpha_X L_X + \beta_X L_C) \left(\mathbb{E}_{\mu}(|X|)^2 - \bar{x}^2 \right)$$

$$+ (\alpha_C L_X + \beta_C L_C) \left(\mathbb{E}_{\mu}(|C|)^2 - \bar{c}^2 \right) - \lambda H(\bar{z}).$$

$$(2.3)$$

Moreover, for all $(z_i)_{1 \leq i \leq N} \in \mathbb{R}^{2N}$, by denoting $H: (z_1, \dots, z_N) \mapsto H(z_i)$,

$$\mathcal{L}^{i,N}H(z_1,\dots,z_N) \le B + (\alpha_X L_X + \beta_X L_C) \left(\left(\frac{1}{N} \sum_{j=1}^N |x_j| \right)^2 - x_i^2 \right) + (\alpha_C L_X + \beta_C L_C) \left(\left(\frac{1}{N} \sum_{j=1}^N |c_j| \right)^2 - c_i^2 \right) - \lambda H(z_i), \tag{2.4}$$

with

$$\alpha_X = \frac{\gamma}{2} + \frac{1}{2}, \quad \beta_X = \frac{17}{2}, \quad \alpha_C = \frac{1}{16}, \quad \beta_C = \frac{1}{2} + \frac{1}{32}.$$

We refer to H as a Lyapunov function, as it ensures that the processes tend to come back to a compact set.

We refer to Appendix A.3 for the proof of this lemma and of the following Proposition.

Proposition 2.3. We have

$$\mathcal{L}^{N}\left(\frac{1}{N}\sum_{i=1}^{N}H\left(Z_{t}^{i,N}\right)\right) \leq B - \lambda\left(\frac{1}{N}\sum_{i=1}^{N}H\left(Z_{t}^{i,N}\right)\right),\tag{2.5}$$

A direct consequence of (2.3) is

$$\mathbb{E}H\left(\bar{Z}_{t}^{i}\right) \leq \mathbb{E}H\left(\bar{Z}_{0}^{i}\right) + \int_{0}^{t} \left(B - \lambda \mathbb{E}H\left(\bar{Z}_{s}^{i}\right)\right) ds,\tag{2.6}$$

and a consequence of (2.5) is

$$\left(\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}H\left(Z_{t}^{i,N}\right)\right)\leq\left(\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}H\left(Z_{0}^{i,N}\right)\right)+\int_{0}^{t}\left(B-\lambda\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}H\left(Z_{s}^{i,N}\right)\right)ds. \tag{2.7}$$

From (2.7) we obtain bounds on the moments of $\left|X_t^{i,N}\right|^2$ and $\left|C_t^{i,N}\right|^2$, and from (2.6) Proposition 1.7 on the second moments of \bar{X}_t^i and \bar{C}_t^i . The proof is given in Appendix A.3. It also yields the following result

Lemma 2.4. Provided the interaction kernels satisfy (2.2), and that $\mathbb{E}(|\bar{X}_0|^2) < +\infty$ and $\mathbb{E}(|\bar{C}_0|^2) < +\infty$, then there exists $\mathcal{C}_{init,2}$ such that for all $t \geq 0$

$$\mathbb{E}\left(|\bar{X}_t|^2 + |\bar{C}_t|^2\right) \le \mathcal{C}_{init,2}.$$

From now on, we consider $\lambda > 0$ satisfying (2.2) (and use the *a priori* bounds $L_{X,\max}$ and $L_{C,\max}$ to ensure the existence of such a λ).

2.3 Modification of the function

Let $C_{init,exp} > 0$, $\tilde{a} > 0$ and consider an initial measure μ_0 on \mathbb{R}^2 which satisfies $\mathbb{E}_{\mu_0}\left(e^{\tilde{a}(|X|+|C|)}\right) \leq C_{init,exp}$, where (X,C) is distributed according to μ_0 .

For technical reasons, we need a greater restoring force by the Lyapunov function than the one given in Lemma 2.2. We thus modify it to obtain estimates such as (2.14) and (2.18) below.

Let a>0, such that $a\leq \tilde{a}/\left(4\sqrt{2}\max\left(\sqrt{\gamma},1\right)\right)$. This choice of a is only necessary for further Propositions and Lemmas, in Section 3.

Let us consider for all $z \in \mathbb{R}^2$,

$$\tilde{H}(z) = \int_0^{H(z)} \exp\left(a\sqrt{u}\right) du = \frac{2}{a^2} \exp\left(a\sqrt{H(z)}\right) \left(a\sqrt{H(z)} - 1\right) + \frac{2}{a^2}.$$
 (2.8)

Direct calculations yield the following technical lemma.

Lemma 2.5. We have, for all $z \in \mathbb{R}^2$

$$H(z) \exp\left(a\sqrt{H(z)}\right) \ge \tilde{H}(z) \ge \exp\left(a\sqrt{H(z)}\right) - \frac{2}{a^2} \left(\exp\left(\frac{a^2}{2}\right) - 1\right),$$
 (2.9)

$$\frac{2}{a}\sqrt{H(z)}\exp\left(a\sqrt{H(z)}\right) \geq \tilde{H}(z) \geq \frac{1}{a}\sqrt{H(z)}\exp\left(a\sqrt{H(z)}\right) - \frac{1}{a^2}\left(e-2\right), \tag{2.10}$$

$$\tilde{H}(z) \ge H(z). \tag{2.11}$$

We may calculate, using Lemma 2.1 and Equation (2.3)

$$\mathcal{L}_{\mu}\left(\tilde{H}\right) = \exp\left(a\sqrt{H}\right) \mathcal{L}_{\mu}H + \frac{1}{2} \frac{a}{2\sqrt{H}} \exp\left(a\sqrt{H}\right) \left(|\sigma_{X}\partial_{X}H|^{2} + |\sigma_{C}\partial_{C}H|^{2}\right) \\
\leq \exp\left(a\sqrt{H}\right) \left(B + (\alpha_{X}L_{X} + \beta_{X}L_{C}) \mathbb{E}_{\mu}(|X|)^{2} + (\alpha_{C}L_{X} + \beta_{C}L_{C}) \mathbb{E}_{\mu}(|C|)^{2} - \lambda H\right) \\
+ \frac{1}{2} \max\left(\sigma_{X}^{2}, \sigma_{C}^{2}\right) \max\left(\gamma, 1\right) a\sqrt{H} \exp\left(a\sqrt{H}\right) \\
\leq \exp\left(a\sqrt{H}\right) \left(B + \frac{\left(\frac{1}{2}\max\left(\sigma_{X}^{2}, \sigma_{C}^{2}\right)\max\left(\gamma, 1\right)\right)^{2} a^{2}}{2\lambda} + (\alpha_{X}L_{X} + \beta_{X}L_{C}) \mathbb{E}_{\mu}(|X|)^{2} \\
+ (\alpha_{C}L_{X} + \beta_{C}L_{C}) \mathbb{E}_{\mu}(|C|)^{2} - \frac{\lambda}{2}H\right), \tag{2.13}$$

where for this last inequality we used Young's inequality

$$\frac{1}{2}\max\left(\sigma_X^2,\sigma_C^2\right)\max\left(\gamma,1\right)a\sqrt{H} \leq \frac{\lambda}{2}H + \frac{\left(\frac{1}{2}\max\left(\sigma_X^2,\sigma_C^2\right)\max\left(\gamma,1\right)\right)^2a^2}{2\lambda}.$$

Notice that (2.13) ensures that this new Lyapunov function also tends to bring back particles into a compact set, and at an even greater rate. This new rate $H\exp(\sqrt{H})$ however comes at a cost: the initial condition must have a finite exponential moment, and no longer just have a finite second moment. First, by Lemma 2.4, $\mathbb{E}(\bar{X}_t)^2 + \mathbb{E}(\bar{C}_t)^2 \leq \mathcal{C}_{init,2}$. Furthermore, the function $h\mapsto \exp\left(a\sqrt{h}\right)\left(B-\frac{\lambda}{4}h\right)$ is bounded from above for $h\geq 0$. We therefore obtain from (2.13) the existence of \tilde{B} such that

$$\mathcal{L}_{\bar{\mu}_{t}}\left(\tilde{H}\left(\bar{Z}_{t}^{i}\right)\right) \leq \tilde{B} - \frac{\lambda}{4}\left(H\left(\bar{Z}_{t}^{i}\right)\exp\left(a\sqrt{H\left(\bar{Z}_{t}^{i}\right)}\right)\right) \tag{2.14}$$

$$\frac{d}{dt}\mathbb{E}\tilde{H}\left(\bar{Z}_{t}^{i}\right) \leq \tilde{B} - \frac{\lambda}{4}\mathbb{E}\left(H\left(\bar{Z}_{t}^{i}\right)\exp\left(a\sqrt{H\left(\bar{Z}_{t}^{i}\right)}\right)\right) \tag{2.15}$$

and
$$\frac{d}{dt}\mathbb{E}\tilde{H}\left(\bar{Z}_{t}^{i}\right) \leq \tilde{B} - \frac{\lambda}{4}\mathbb{E}\tilde{H}\left(\bar{Z}_{t}^{i}\right),$$
 (2.16)

where for this last inequality, we used (2.9). While (2.14) and (2.15) will be useful in ensuring a sufficient restoring force, Equation (2.16) gives us a uniform in time bound on $\mathbb{E}\tilde{H}\left(\bar{Z}_t^i\right)$, provided we have an initial bound. These inequalities are to be understood in the sense of SDEs, where (2.16) should for instance be rigorously written

$$\mathbb{E}\tilde{H}\left(\bar{Z}_{t}^{i}\right) \leq \mathbb{E}\tilde{H}\left(\bar{Z}_{0}^{i}\right) + \int_{0}^{t} \left(\tilde{B} - \frac{\lambda}{4}\mathbb{E}\tilde{H}\left(\bar{Z}_{s}^{i}\right)\right) ds.$$

Now, for the system of particles, we have, using (2.13), $\forall i, \forall x_i, v_i \in \mathbb{R}^d$

$$\mathcal{L}^{N}\tilde{H}\left(z_{i}\right) \leq \exp\left(a\sqrt{H\left(z_{i}\right)}\right) \left(\tilde{B} + \left(\alpha_{X}L_{X} + \beta_{X}L_{C}\right)\left(\frac{1}{N}\sum_{j=1}^{N}\left|z_{j}\right|\right)^{2} + \left(\alpha_{C}L_{X} + \beta_{C}L_{C}\right)\left(\frac{1}{N}\sum_{j=1}^{N}\left|c_{j}\right|\right)^{2} - \frac{\lambda}{2}H\left(z_{i}\right)\right).$$

Summing over $i \in \{1, ..., N\}$, we may calculate

$$(\alpha_{X}L_{X} + \beta_{X}L_{C}) \sum_{j=1}^{N} \left(\frac{\sum_{j=1}^{N} |x_{j}|}{N}\right)^{2} \sum_{i=1}^{N} \frac{\exp\left(a\sqrt{H\left(z_{i}\right)}\right)}{N} - \frac{\lambda}{16} \sum_{i=1}^{N} \frac{H\left(z_{i}\right) \exp\left(a\sqrt{H\left(z_{i}\right)}\right)}{N}$$

$$\leq \frac{\lambda}{16} \left(\sum_{i,j=1}^{N} \frac{H\left(z_{i}\right)}{N} \frac{\exp\left(a\sqrt{H\left(z_{j}\right)}\right)}{N} - \sum_{i=1}^{N} \frac{H\left(z_{i}\right) \exp\left(a\sqrt{H\left(z_{i}\right)}\right)}{N}\right)$$

$$\leq 0.$$

Here, we used Lemma 2.1, the fact that $\forall x,y \geq 0, xe^{\sqrt{y}} + ye^{\sqrt{x}} - xe^{\sqrt{x}} - ye^{\sqrt{y}} = (e^{\sqrt{x}} - e^{\sqrt{y}})(y-x) \leq 0$ and assumed

$$(\alpha_X L_X + \beta_X L_C) \le \frac{\gamma \lambda}{64}.$$

Likewise,

$$\left(\alpha_{C}L_{X}+\beta_{C}L_{C}\right)\sum_{j=1}^{N}\left(\frac{\sum_{j=1}^{N}\left|c_{j}\right|}{N}\right)^{2}\sum_{i=1}^{N}\frac{\exp\left(a\sqrt{H\left(z_{i}\right)}\right)}{N}\\ -\frac{\lambda}{16}\sum_{i=1}^{N}\frac{H\left(z_{i}\right)\exp\left(a\sqrt{H\left(z_{i}\right)}\right)}{N}\leq0,$$

provided

$$(\alpha_C L_X + \beta_C L_C) \le \frac{\lambda}{64}.$$

There is therefore a constant, which for the sake of clarity we will also denote B (as we may take the maximum of the previous constants), such that we get

$$\mathcal{L}^{i,N}\tilde{H}(Z_t^{i,N}) \leq \tilde{B} + (\alpha_X L_X + \beta_X L_C) \left(\frac{\sum_{j=1}^N |X_t^{j,N}|}{N}\right)^2 \exp\left(a\sqrt{H\left(Z_t^{i,N}\right)}\right)$$

$$+ (\alpha_C L_X + \beta_C L_C) \left(\frac{\sum_{j=1}^N |C_t^{j,N}|}{N}\right)^2 \exp\left(a\sqrt{H\left(Z_t^{i,N}\right)}\right)$$

$$- \frac{\lambda}{4} H\left(Z_t^{i,N}\right) \exp\left(a\sqrt{H\left(Z_t^{i,N}\right)}\right), \tag{2.17}$$

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$$\mathcal{L}^{N}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{H}(Z_{t}^{i,N})\right) \leq \tilde{B} - \frac{\lambda}{4}\left(\frac{1}{N}\sum_{i=1}^{N}H(Z_{t}^{i,N})\exp\left(a\sqrt{H\left(Z_{t}^{i,N}\right)}\right)\right),\tag{2.18}$$

and

$$\mathcal{L}^{N}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{H}(Z_{t}^{i,N})\right) \leq \tilde{B} - \frac{\lambda}{4}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{H}(Z_{t}^{i,N})\right). \tag{2.19}$$

Once again, (2.17) and (2.18) will be useful in ensuring a sufficient restoring force, and (2.19) yields a uniform in time bound on the expectation of $\tilde{H}(Z_t^{i,N})$, since by exchangeability of the particles, $\mathbb{E}\left(\frac{1}{N}\sum_{j=1}^N \tilde{H}(Z_t^{j,N})\right) = \mathbb{E}\left(\tilde{H}(Z_t^{i,N})\right)$.

2.4 Parameters

We start by fixing the values of some parameters. The somewhat intricate expressions in this section are dictated by the computations arising in the proofs later on. They are somewhat roughly chosen and far from optimal as we only wish to convey the fact that every constant is explicit. On first reading, the exact choice of parameters can and should be skipped, as they are only meant to satisfy Lemma 2.6, which is the crucial Lemma of this subsection.

Recall α_X , β_X , α_C and β_C given in Lemma 2.2. a>0 is fixed from the last Subsection and the definition of \tilde{H} , and λ and \tilde{B} are obtained from the same Subsection.

Given any $\eta > 4$ and $\tilde{\delta} > 0$, consider the following set of parameters

$$\begin{split} \delta = & (1+\tilde{\delta})\frac{1+L_{X,\max}}{1-L_{C,\max}}, \ R_0 = \sqrt{\frac{1280\tilde{B}}{\lambda \min(\gamma,1)}}, \ R = \sqrt{1+\delta^2}R_0, \\ \mathcal{C}_{f,1} = & 16\left(\left(\gamma + a\left(\beta + \frac{\alpha}{\delta}\right)\sqrt{2\max\left(\gamma,1\right)}\right)\frac{e^{a^2/2} - 1}{a^2} + \sqrt{2\max\left(\gamma,1\right)}\left(\sqrt{\gamma} + \frac{1}{\delta}\right)\left(e - 2\right)\right), \\ \mathcal{C}_{f,2} = & 4\left(\gamma + \left(a\left(\beta + \frac{\alpha}{\delta}\right) + 2a^2\left(\sqrt{\gamma} + \frac{1}{\delta}\right)\right)\sqrt{2\max\left(\gamma,1\right)}\right), \\ c = & \min\left\{\frac{2\tilde{B}}{\eta}, \frac{\lambda}{160}\frac{\eta - 4}{\eta}, \\ \frac{\min\left(\frac{\sigma_X}{\sqrt{\pi}R}, 1 - L_{C,\max} - \frac{1+L_{X,\max}}{\delta}\right)}{2(1+\eta)} \right. \\ & \times \exp\left(-\frac{1}{4\sigma_X^2}\left(1 + \delta\gamma + L_{X,\max} + \delta L_{C,\max} + \left(\mathcal{C}_{f,1} + \mathcal{C}_{f,2}\right)\sigma_X^2\right)R^2\right)\right\}, \\ \epsilon = & \frac{\eta c}{2\tilde{B}}, \phi_{\min} = \exp\left(-\frac{1}{4\sigma_X^2}\left(1 + \delta\gamma + L_{X,\max} + \delta L_{C,\max} + \left(\epsilon\mathcal{C}_{f,1} + \mathcal{C}_{f,2}\right)\sigma_X^2\right)R^2\right), \\ \mathcal{C}_1 = & \frac{1}{\min\left(\delta,1\right)}\frac{2}{\phi_{\min}}\max\left(\frac{16(1+\delta^2)}{\epsilon\min\left(\gamma,1\right)}, 1\right), \mathcal{C}_2 = & \frac{1}{\min\left(\delta^2,1\right)}\frac{2}{\phi_{\min}}\max\left(\frac{16(1+\delta^2)}{\epsilon\min\left(\gamma,1\right)}, 1\right), \\ \mathcal{C}_2 = & \frac{2}{\phi_{\min}}\max\left(1, \frac{4}{\epsilon}\max\left(\sqrt{\frac{1}{\gamma}}, 1\right)\right). \end{split}$$

We define f as follows

$$f(r) = \int_0^{r \wedge R} \phi(s)g(s)ds,$$

$$\phi(r) = \exp\left(-\frac{1}{4\sigma_X^2} \left(1 + \delta\gamma + L_X + \delta L_C + (\epsilon C_{f,1} + C_{f,2})\sigma_X^2\right)r^2\right),$$
(2.20)

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$$\Phi(s) = \int_0^s \phi(u) du,$$

$$g(r) = 1 - \frac{c + 2\epsilon \tilde{B}}{\sigma_Y^2} \int_0^r \Phi(s) \phi(s)^{-1} ds.$$

Assume furthermore that L_X and L_C , the Lipschitz constants, satisfy

$$L_X \le \min\left(\frac{\lambda}{128\mathcal{C}_z}, \frac{\lambda a}{512\epsilon\mathcal{C}_z}, \frac{c}{2\mathcal{C}_1}\right) \quad \text{and} \quad L_C \le \min\left(\frac{\lambda}{128\delta\mathcal{C}_z}, \frac{\lambda a}{512\epsilon\delta\mathcal{C}_z}, \frac{c}{2\delta\mathcal{C}_1}\right), \quad (2.21)$$

$$\alpha_X L_X + \beta_X L_C \le \frac{\gamma \lambda}{128}$$
 and $\alpha_C L_X + \beta_C L_C \le \frac{\lambda}{128}$, (2.22)

$$\frac{L_X}{8} + L_C \left(2 + \frac{1}{8} \right) < 1 - \frac{\lambda}{2}. \tag{2.23}$$

Notice how the bounds on L_X and L_C depend on c. This is one of the reasons why we use the *a priori* bounds $L_X \in [0, L_{X, \max}]$ and $L_C \in [0, L_{C, \max}]$ given in the assumptions of Theorem 1.4: they allow us to bound c and δ independently of L_C and L_X . We are thus able to begin by choosing acceptable values for those parameters, before then giving upper bounds on L_X and L_C . The condition of taking L_X and L_C small enough (the condition $L_X < c^K$ and $L_C < c^K$ for a well chosen c^K , given in Theorem 1.4) is necessary to satisfy the conditions of (2.21), (2.22) and (2.23).

We quickly mention that the constants \mathcal{C}_1 , \mathcal{C}_2 and \mathcal{C}_z above come from Lemma 2.7 later. We gather some properties required in the calculations of the proof of Theorem 1.4 in the following lemma. Again, these properties are the ones motivating the choice of parameters

Lemma 2.6. The set of parameters given in Subsection 2.4 satisfy

• f is C^2 on (0,R) such that $f'_+(0)=1$ and $f'_-(R)>0$, and constant on $[R,\infty)$. Moreover, f is non-negative, non-decreasing, and concave, and for all $s\geq 0$,

$$\min\left(s,R\right)f_{-}'\left(R\right) \leq f\left(s\right) \leq \min\left(s,f\left(R\right)\right) \leq \min\left(s,R\right).$$

- For all $r \in [0, R]$, $\phi(r) \ge \phi_{\min}$ and $g(r) \ge \frac{1}{2}$.
- · We have the conditions

$$\begin{split} \frac{2}{f}'(R) &\geq \exp\left(-\frac{1}{4\sigma_X^2}\left(1 + \delta\gamma + L_X + \delta L_C + (\epsilon\mathcal{C}_{f,1} + \mathcal{C}_{f,2})\,\sigma_X^2\right)R^2\right), \\ 2c + 4\epsilon\tilde{B} &\leq \left(1 - L_C - \frac{1 + L_X}{\delta}\right) \min_{r \in (0,R]} \frac{f'(r)r}{f(r)}, \\ c &\leq \frac{\lambda}{160} \frac{\frac{80\epsilon\tilde{B}}{\lambda}}{1 + \frac{80\epsilon\tilde{B}}{\lambda}}, \quad \frac{1 + L_X}{1 - L_C} < \delta \quad \text{and} \quad \epsilon \leq 1. \end{split}$$

The proof of this lemma is done in Appendix A.4.

2.5 Control of the usual distances

As explained previously, we consider a modified semi-metric. For $z=(x,c)\in\mathbb{R}^2$ and $z'=(x',c')\in\mathbb{R}^2$, define

$$r(z, z') = r(x, c, x', c') = |x - x'| + \delta|c - c'|, \tag{2.24}$$

where δ is given in Subsection 2.4, and let $\rho((z_j,z_j')_{1 \le j \le N})$ be defined as follows

$$\rho\left(\left(z_{j}, z_{j}'\right)_{1 \leq j \leq N}\right) = \frac{1}{N} \sum_{i=1}^{N} f\left(r\left(z_{i}, z_{i}'\right)\right) G^{i}\left(\left(z_{j}, z_{j}'\right)_{j}\right), \tag{2.25}$$

where for each $i \in \{1, \dots, N\}$,

$$G^{i}\left(\left(z_{j}, z_{j}^{\prime}\right)_{j}\right) = 1 + \epsilon \tilde{H}\left(z_{i}\right) + \epsilon \tilde{H}\left(z_{i}^{\prime}\right) + \frac{\epsilon}{N} \sum_{j=1}^{N} \tilde{H}\left(z_{j}\right) + \frac{\epsilon}{N} \sum_{j=1}^{N} \tilde{H}\left(z_{j}^{\prime}\right). \tag{2.26}$$

An immediate corollary of the definition and properties of H is that ρ is a quantity on \mathbb{R}^{4N} which controls the usual L^1 and L^2 distances.

Lemma 2.7. The constants $C_1, C_2, C_z > 0$, given in Subsection 2.4, are such that for all $z = (x, c) \in \mathbb{R}^2$ and $z' = (x', c') \in \mathbb{R}^2$

(i)
$$||z-z'||_1 \leq C_1 f(r(z,z')) \left(1 + \epsilon \tilde{H}(z) + \epsilon \tilde{H}(z')\right)$$

(ii)
$$||z-z'||_2^2 \le C_2 f\left(r\left(z,z'\right)\right) \left(1+\epsilon \tilde{H}(z)+\epsilon \tilde{H}(z')\right)$$

(iii)
$$||z-z'||_1 \leq C_z f(r(z,z')) \left(1 + \epsilon \sqrt{H(z)} + \epsilon \sqrt{H(z')}\right)$$
.

The proof of this lemma is postponed to Appendix A.5.

3 Proof of Theorem 1.4 in the case $\sigma_X > 0$

Let $\xi>0$ be a parameter destined for vanishing, and let $\varphi_{\rm sc}:\mathbb{R}^+\mapsto\mathbb{R}^+$ and $\varphi_{\rm rc}:\mathbb{R}^+\mapsto\mathbb{R}^+$ be two Lipschitz continuous functions such that

$$\forall x, \quad \varphi_{\rm sc}^2(x) + \varphi_{\rm rc}^2(x) = 1,$$

$$\varphi_{\rm rc}(x) = 1 \text{ if } \xi \le x \le R,$$

$$\varphi_{\rm rc}(x) = 0 \text{ if } x \le \frac{\xi}{2} \text{ or } x \ge R + \xi.$$

$$(3.1)$$

Intuitively, φ_{rc} represents the region of space in which we consider a reflection coupling, and φ_{sc} the one in which we consider a synchronous coupling. In reality, we would like to consider φ_{sc} and φ_{rc} indicator functions of the regions of space. However, we need to consider a Lipschitz approximation of indicator functions to ensure continuity (to apply Itô's calculus) and the strong existence and uniqueness of the stochastic processes. We thus simultaneously construct the following solutions, for $1 \le i \le N$

$$\begin{cases} dX_t^{i,N} = & (X_t^{i,N} - (X_t^{i,N})^3 - C_t^{i,N} - \alpha)dt + \frac{1}{N} \sum_{j=1}^N K_X(Z_t^{i,N} - Z_t^{j,N})dt \\ & + \sigma_X \varphi_{\text{sc}} \left(|X_t^{i,N} - \bar{X}_t^{i}| \right) dB_t^{i,sc,X} + \sigma_X \varphi_{\text{rc}} \left(|X_t^{i,N} - \bar{X}_t^{i}| \right) dB_t^{i,rc,X} \\ dC_t^{i,N} = & (\gamma X_t^{i,N} - C_t^{i,N} + \beta)dt + \frac{1}{N} \sum_{j=1}^N K_C(Z_t^{i,N} - Z_t^{j,N})dt + \sigma_C dB_t^{i,C}, \end{cases}$$

and

$$\begin{cases} d\bar{X}_{t}^{i} = & (\bar{X}_{t}^{i} - (\bar{X}_{t}^{i})^{3} - \bar{C}_{t}^{i} - \alpha)dt + K_{X} * \bar{\mu}_{t}(\bar{Z}_{t}^{i})dt \\ & + \sigma_{X}\varphi_{\text{sc}}\left(|X_{t}^{i,N} - \bar{X}_{t}^{i}|\right)dB_{t}^{i,sc,X} - \sigma_{X}\varphi_{\text{rc}}\left(|X_{t}^{i,N} - \bar{X}_{t}^{i}|\right)dB_{t}^{i,rc,X} \\ d\bar{C}_{t}^{i} = & (\gamma\bar{X}_{t}^{i} - \bar{C}_{t}^{i} + \beta)dt + K_{C} * \bar{\mu}_{t}(\bar{Z}_{t}^{i})dt + \sigma_{C}dB_{t}^{i,C}, \end{cases}$$

where $(B^{i,sc,X})_i$ and $(B^{i,rc,X})_i$ are independent Brownian motions (also independent of $(B^{i,C})_i$). Notice that we consider a symmetric coupling on the dynamics of C. By Levy's characterization of Brownian motion, using (3.1), we thus construct a solution of (1.1) and N independent copies of a solution of (1.2).

3.1 Main proof and results

Proposition 3.1. We denote $r_t^i=r(Z_t^{i,N},\bar{Z}_t^i)$ and $G_t^i=G^i((Z_t^{j,N})_j,(\bar{Z}_t^j)_j)$. For all $c\in\mathbb{R}$, for each $i\in\{1,\dots,N\}$, we have

$$d(e^{ct}f(r_t^i)G_t^i) \le e^{ct}K_t^i dt + dM_t^i, \tag{3.2}$$

where M_t^i is a continuous local martingale and K_t^i can be written as

$$K_t^i = \tilde{K}_t^i + I_t^{1,i} + I_t^{2,i} + I_t^{3,i}. \tag{3.3}$$

We define \tilde{K}_t^i , $I_t^{1,i}$, $I_t^{2,i}$ and $I_t^{3,i}$ as follows

$$\begin{split} \hat{K}_{t}^{i} &= G_{t}^{i} \Big[2cf(r_{t}^{i}) + \frac{1}{2} f''(r_{t}^{i}) \left(2\sigma_{X}^{2} \varphi_{rc} \left(|X_{t}^{i,N} - \bar{X}_{t}^{i}| \right)^{2} \right) \\ &+ f'(r_{t}^{i}) \left((1 + \gamma \delta + L_{X} + \delta L_{C}) |X_{t}^{i,N} - \bar{X}_{t}^{i}| - |(X_{t}^{i,N})^{3} - (\bar{X}_{t}^{i})^{3}| \right. \\ &+ (1 + L_{X} + \delta L_{C} - \delta) |C_{t}^{i,N} - \bar{C}_{t}^{i}| + (\epsilon C_{f,1} + C_{f,2}) \sigma_{X}^{2} \varphi_{rc} \left(|X_{t}^{i,N} - \bar{X}_{t}^{i}| \right)^{2} r_{t}^{i} \right) \Big] \\ &+ \epsilon f(r_{t}^{i}) \left(4\tilde{B} - \frac{\lambda}{16} \tilde{H}(\bar{Z}_{t}^{i}) - \frac{\lambda}{16} \tilde{H}(Z_{t}^{i,N}) - \frac{\lambda}{16N} \sum_{j=1}^{N} \tilde{H}(\bar{Z}_{t}^{j}) - \frac{\lambda}{16N} \sum_{j=1}^{N} \tilde{H}(Z_{t}^{j,N}) \right), \end{split}$$
(3.4)
$$I_{t}^{1,i} &= G_{t}^{i} f'(r_{t}^{i}) \left(\left| \frac{1}{N} \sum_{j=1}^{N} K_{X}(\bar{Z}_{t}^{i} - \bar{Z}_{t}^{j}) - K_{X} * \bar{\mu}_{t}(\bar{Z}_{t}^{i}) \right| \right) \\ &+ \delta G_{t}^{i} f'(r_{t}^{i}) \left(\left| \frac{1}{N} \sum_{j=1}^{N} K_{C}(\bar{Z}_{t}^{i} - \bar{Z}_{t}^{j}) - K_{C} * \bar{\mu}_{t}(\bar{Z}_{t}^{i}) \right| \right) \\ &+ \delta G_{t}^{i} f'(r_{t}^{i}) \left(\left| \frac{1}{N} \sum_{j=1}^{N} K_{C}(\bar{Z}_{t}^{i} - \bar{Z}_{t}^{j}) - K_{C} * \bar{\mu}_{t}(\bar{Z}_{t}^{i}) \right| \right) \\ &- cf(r_{t}^{i}) G_{t}^{i} - \epsilon f(r_{t}^{i}) \left[\frac{\lambda}{16} H(\bar{Z}_{t}^{i}) \exp\left(a \sqrt{H(\bar{Z}_{t}^{i})} \right) + \frac{\lambda}{16} H(Z_{t}^{i,N}) \exp\left(a \sqrt{H(Z_{t}^{i,N})} \right) \right] \\ &- \epsilon f(r_{t}^{i}) \left[\frac{\lambda}{16N} \sum_{j=1}^{N} H(\bar{Z}_{t}^{j}) \exp\left(a \sqrt{H(\bar{Z}_{t}^{j,N})} \right) \right] \\ &+ \frac{\lambda}{16N} \sum_{j=1}^{N} H(Z_{t}^{j,N}) \exp\left(a \sqrt{H(Z_{t}^{i,N})} \right) \right] \\ &+ (\alpha_{C} L_{X} + \beta_{X} L_{C}) \left(\frac{\sum_{j=1}^{N} |X_{t}^{j,N}|}{N} \right)^{2} \exp\left(a \sqrt{H(Z_{t}^{i,N})} \right) \\ &- \frac{\lambda}{16} H(Z_{t}^{i,N}) \exp\left(a \sqrt{H(Z_{t}^{i,N})} \right) - \frac{\lambda}{16N} \sum_{j=1}^{N} H(Z_{t}^{j,N}) \exp\left(a \sqrt{H(Z_{t}^{i,N})} \right) \right). \end{split}$$
 (3.6)

We need to control $\mathbb{E}(G_t^i)$. This control is a consequence of Lyapunov's properties on \tilde{H} and the initial assumption of the Theorem 1.4. A proof is given in Appendix A.6.

Lemma 3.2. There exist $C_{G,1}$ and $C_{G,2}$, such that for each $i \leq N$, for all t > 0, we have

$$\mathbb{E}(G_t^i) \leq \mathcal{C}_{G,1}$$
 and $\mathbb{E}[{(G_t^i)}^2] \leq \mathcal{C}_{G,2}.$

The decomposition given in the Proposition 3.1 is true for all $c \in \mathbb{R}$. To control exactly the behavior of each term, we will now consider c, defined in Subsection 2.4.

Each term given in Proposition 3.1 will be controlled differently. The following lemmas summarize it. The first term, \tilde{K}^i_t , contains the various behaviors we have previously identified: we deal with it either through a synchronous coupling (when the deterministic drift is contracting) or through a reflection coupling (notice the second derivative f'' which will provide contraction provided f is sufficiently concave). Finally, notice the effect of the Lyapunov function \tilde{H} which yields a restoring force.

Lemma 3.3. With the parameters and functions given in Subsection 2.4, for each $i \leq N$, for all t > 0,

$$\mathbb{E}\tilde{K}_{t}^{i} \leq \xi \left(2 + \delta \gamma + L_{X} + \delta L_{C} - L_{C} - \frac{1 + L_{X}}{\delta}\right) \mathbb{E}G_{t}^{i}.$$
(3.8)

The interaction term $\frac{1}{N}\sum_j K_X(Z_t^{j,N}-Z_t^{i,N})-K_X*\bar{\mu}_t(\bar{Z}_t^i)$ can be decomposed into the following two parts: $\frac{1}{N}\sum_j K_X(\bar{Z}_t^j-\bar{Z}_t^i)-K_X*\bar{\mu}_t(\bar{Z}_t^i)$ and $\frac{1}{N}\sum_j [K_X(Z_t^{j,N}-Z_t^{i,N})-K_X(\bar{Z}_t^j-\bar{Z}_t^i)]$. The first part, which is in $I_t^{1,i}$, is dealt with using some form of the law of large numbers in a similar way to what has been done in the proof of Theorem 1.3.

Lemma 3.4. With the parameters and functions given in Subsection 2.4, for each $i \leq N$, for all t > 0,

$$\mathbb{E}(I_t^{1,i}) \le 4\sqrt{\frac{\mathcal{C}_{init,2}\mathcal{C}_{G,2}}{N}}(L_X + L_C),\tag{3.9}$$

where $C_{G,2}$ is defined in Lemma 3.2 and $C_{init,2}$ is defined in Lemma 2.4.

 $I_t^{2,i}$ contains the leftovers of this decomposition and some of the additional terms of the Lyapunov function.

Lemma 3.5. With the parameters and functions given in Subsection 2.4, for all t > 0,

$$\frac{1}{N} \sum_{i=1}^{N} I_t^{2,i} \le 0. {(3.10)}$$

Finally, $I_t^{3,i}$ deals with the non-linearity appearing in the dynamics of the Lyapunov function, and will be non-positive for values of L_X and L_C sufficiently small. It is also here we justify adding the last two terms in (2.26).

Lemma 3.6. With the parameters and functions given in Subsection 2.4, for each $i \leq N$, for all t > 0,

$$I_t^{3,i} \le 0. {(3.11)}$$

Proof of Theorem 1.4. With these four Lemmas, we can calculate

$$\begin{split} &\frac{1}{N} \sum_{i=1}^{N} \mathbb{E} K_{t}^{i} = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \tilde{K}_{t}^{i} + \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} I_{t}^{1,i} + \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} I_{t}^{2,i} + \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} I_{t}^{3,i} \\ &\leq \xi \left(2 + \delta \gamma + L_{X} + \delta L_{C} - L_{C} - \frac{1 + L_{X}}{\delta} \right) \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} G_{t}^{i} + 4 \sqrt{\frac{C_{init,2}C_{G,2}}{N}} (L_{X} + L_{C}) \end{split}$$

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Since by Lemma 3.2, we have $\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}G_{t}^{i}\leq\mathcal{C}_{G,1}$, we obtain

$$\frac{1}{N} \sum_{i=1}^{N} \mathbb{E}K_t^i \le \xi A + (L_X + L_C) \frac{B}{\sqrt{N}}$$

where A and B are constants.

For all initial couplings such that $\mathbb{E}\rho\left((Z_0^{j,N},\bar{Z}_0^j)_{1\leq j\leq N}\right)<\infty$, by taking the expectation of (3.2) along a sequence of increasing localizing stopping times, we have thanks to Fatou's lemma

$$e^{ct}\mathbb{E}\left(\rho\left((Z_t^{j,N},\bar{Z}_t^j)_{1\leq j\leq N}\right)\right)\leq \mathbb{E}\left(\rho\left((Z_0^{j,N},\bar{Z}_0^j)_{1\leq j\leq N}\right)\right)+\xi A\int_0^t e^{cs}ds$$
$$+\left(L_X+L_C\right)\frac{B}{\sqrt{N}}\int_0^t e^{cs}ds.$$

We obtain

$$\mathbb{E}\left(\rho\left((Z_t^{j,N}, \bar{Z}_t^j)_{1 \le j \le N}\right)\right) \le \mathbb{E}\left(\rho\left((Z_0^{j,N}, \bar{Z}_0^j)_{1 \le j \le N}\right)\right) e^{-ct} + \frac{\xi A}{c} \left(1 - e^{-ct}\right) + \frac{(L_X + L_C)B}{c} \frac{1}{\sqrt{N}} \left(1 - e^{-ct}\right).$$

By using the exchangeability of the particles, we have $\mathbb{E}\left(\rho\left((Z_t^{j,N},\bar{Z}_t^j)_{1\leq j\leq N}\right)\right)=\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^N f(r_t^i)G_t^i\right)=\mathbb{E}\left(\frac{1}{k}\sum_{i=1}^k f(r_t^i)G_t^i\right)$ for all $k\in\mathbb{N}$. Then

$$\mathbb{E}\left(\sum_{i=1}^{k} f(r_t^i) G_t^i\right) = k \mathbb{E}\left(\rho\left((Z_t^{j,N}, \bar{Z}_t^j)_{1 \le j \le N}\right)\right).$$

Let μ_0 be a measure on \mathbb{R}^2 , $\mu_t^{k,N}$ the marginal distribution at time t of the first k neurons $\left((X_t^{1,N},C_t^{1,N}),\ldots,(X_t^{k,N},C_t^{k,N})\right)$ of an N-particle system (1.1) with initial distribution $\mu_0^{\otimes N}$, and $\bar{\mu}_t$ a solution of (1.2) with initial distribution μ_0 . This implies $\mathbb{E}\left(\rho\left((Z_0^{j,N},\bar{Z}_0^j)_{1\leq j\leq N}\right)\right)=0$. By Lemma 2.7, we obtain for the L^1 Wasserstein distance

$$\begin{split} \mathcal{W}_{1}(\mu_{t}^{k,N},\bar{\mu}_{t}^{\otimes k}) &= \inf \left\{ \mathbb{E}[\|Z^{(k)} - \bar{Z}^{(k)}\|_{1}], \mathbb{P}_{Z^{(k)}} = \mu_{t}^{k,N}, \mathbb{P}_{\bar{Z}^{(k)}} = \bar{\mu}_{t}^{\otimes k} \right\} \\ &\leq \inf \left\{ \mathcal{C}_{1} \mathbb{E}\left[\sum_{i=1}^{k} f(r_{t}^{i}) G_{t}^{i}\right], \mathbb{P}_{\left(Z_{t}^{i,N}\right)_{i}} = \mu_{t}^{k,N}, \mathbb{P}_{\left(\bar{Z}_{t}^{i}\right)_{i}} = \bar{\mu}_{t}^{\otimes k} \right\} \\ &\leq \inf \left\{ k\mathcal{C}_{1} \mathbb{E}\left(\rho\left((Z_{t}^{j,N}, \bar{Z}_{t}^{j})_{1 \leq j \leq N}\right)\right), \mathbb{P}_{\left(Z_{t}^{i,N}\right)_{i}} = \mu_{t}^{k,N}, \mathbb{P}_{\left(\bar{Z}_{t}^{i}\right)_{i}} = \bar{\mu}_{t}^{\otimes k} \right\} \\ &\leq \frac{\xi Ak\mathcal{C}_{1}}{c} \left(1 - e^{-ct}\right) + \frac{(L_{X} + L_{C})B\mathcal{C}_{1}}{c} \frac{k}{\sqrt{N}} \left(1 - e^{-ct}\right) \end{split}$$

By taking the limit as $\xi \to 0$ uniformly in time, we obtain the desired result. The same lemma and the same type of calculations yield the result for the L^2 Wasserstein distance

$$\mathcal{W}_2(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k})^2 \le \frac{k}{\sqrt{N}} \mathcal{C}_2 \frac{(L_X + L_C)B}{c}.$$

3.2 Proof of the decomposition

Proof of Proposition 3.1. First, we need to calculate $d(e^{ct}f(r_t^i)G_t^i)$, where we recall

$$r_t^i = |X_t^{i,N} - \bar{X}_t^i| + \delta |C_t^{i,N} - \bar{C}_t^i|$$

and

$$G_t^i = 1 + \epsilon \tilde{H}(\bar{Z}_t^i) + \epsilon \tilde{H}(Z_t^{i,N}) + \frac{\epsilon}{N} \sum_{j=1}^N \tilde{H}(Z_t^{j,N}) + \frac{\epsilon}{N} \sum_{j=1}^N \tilde{H}(\bar{Z}_t^j).$$

We have already calculated $d(X_t^{i,N}-\bar{X}_t^i)$ and $d|X_t^{i,N}-\bar{X}_t^i|$ in the case of symmetric coupling in Subsection 1.4 in (1.8). Here, we need to use Ito's formula and usual convergence lemmas, see Lemma A.1 below, to take care of the Brownian term (recall that the coefficient in front of the Brownian vanishes in the vicinity of the singularity of the absolute value). We obtain

$$d|X_t^{i,N} - \bar{X}_t^i| = A_t^X dt + 2\mathrm{sign}(X_t^{i,N} - \bar{X}_t^i)\sigma_X\varphi_{\mathrm{rc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right)dB_t^{i,rc,X},$$

with

$$A_t^X \le |X_t^{i,N} - \bar{X}_t^i| - \left| (X_t^{i,N})^3 - (\bar{X}_t^i)^3 \right| + \left| C_t^{i,N} - \bar{C}_t^i \right| + \left| \frac{1}{N} \sum_{j=1}^N K_X (Z_t^{i,N} - Z_t^{j,N}) - K_X * \bar{\mu}_t(\bar{Z}_t^i) \right|.$$

Likewise, as it has already been calculated in (1.9) in Subsection 1.4,

$$d|C_t^{i,N} - \bar{C}_t^i| = A_t^C dt,$$

with

$$A_t^C \le \gamma \left| X_t^{i,N} - \bar{X}_t^i \right| - |C_t^{i,N} - \bar{C}_t^i| + \left| \frac{1}{N} \sum_{j=1}^N K_C(Z_t^{i,N} - Z_t^{j,N}) - K_C * \bar{\mu}_t(\bar{Z}_t^i) \right|.$$

Now we have

$$dr_t^i = \left(A_t^X + \delta A_t^C\right)dt + 2\mathrm{sign}(X_t^{i,N} - \bar{X}_t^i)\sigma_X\varphi_{\mathrm{rc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right)dB_t^{i,rc,X}$$

and we deduce with Ito's formula

$$df(r_t^i) = f'(r_t^i)dr_t^i + \frac{1}{2}f''(r_t^i)\left(2\sigma_X\varphi_{\rm rc}\left(|X_t^{i,N} - \bar{X}_t^i|\right)\right)^2dt.$$

Finally, for c > 0,

$$d(e^{ct}f(r_t^i)) = ce^{ct}f(r_t^i)dt + e^{ct}df(r_t^i).$$

Then, by Ito's formula,

$$\begin{split} \frac{1}{\epsilon} dG_t^i &= \left(\mathcal{L}_{\bar{\mu}t} \tilde{H}(\bar{Z}_t^i) + \mathcal{L}^N \tilde{H}(Z_t^{i,N}) \right) dt \\ &+ \sigma_X \varphi_{\rm rc} \left(|X_t^{i,N} - \bar{X}_t^i| \right) \left(\partial_X \tilde{H}(Z_t^{i,N}) - \partial_X \tilde{H}(\bar{Z}_t^i) \right) dB_t^{i,rc,X} \\ &+ \sigma_X \varphi_{\rm sc} \left(|X_t^{i,N} - \bar{X}_t^i| \right) \left(\partial_X \tilde{H}(Z_t^{i,N}) + \partial_X \tilde{H}(\bar{Z}_t^i) \right) dB_t^{i,sc,X} \\ &+ \sigma_C \left(\partial_C \tilde{H}(Z_t^{i,N}) + \partial_C \tilde{H}(\bar{Z}_t^i) \right) dB_t^{i,C} \\ &+ \frac{1}{N} \sum_{j=1}^N \left(\mathcal{L}_{\bar{\mu}_t} \tilde{H}(\bar{Z}_t^j) + \mathcal{L}^N \tilde{H}(Z_t^{j,N}) \right) dt \\ &+ \frac{\sigma_X}{N} \sum_{j=1}^N \varphi_{\rm rc} \left(|X_t^{j,N} - \bar{X}_t^j| \right) \left(\partial_X \tilde{H}(Z_t^{j,N}) - \partial_X \tilde{H}(\bar{Z}_t^j) \right) dB_t^{j,rc,X} \\ &+ \frac{\sigma_X}{N} \sum_{j=1}^N \varphi_{\rm sc} \left(|X_t^{j,N} - \bar{X}_t^j| \right) \left(\partial_X \tilde{H}(Z_t^{j,N}) + \partial_X \tilde{H}(\bar{Z}_t^j) \right) dB_t^{j,sc,X} \\ &+ \frac{\sigma_C}{N} \sum_{j=1}^N \left(\partial_C \tilde{H}(Z_t^{j,N}) + \partial_C \tilde{H}(\bar{Z}_t^j) \right) dB_t^{j,C}. \end{split}$$

We finally get

$$\begin{split} d(e^{ct}f(r_t^i)G_t^i) &= G_t^i d(e^{ct}f(r_t^i)) + e^{ct}f(r_t^i)dG_t^i \\ &+ 2\epsilon \left(1 + \frac{1}{N}\right)\sigma_X^2 \varphi_{\text{rc}} \Big(|X_t^{i,N} - \bar{X}_t^i|\Big)^2 \text{sign}(X_t^{i,N} - \bar{X}_t^i) \\ &\qquad \times \Big(\partial_X \tilde{H}(Z_t^{i,N}) - \partial_X \tilde{H}(\bar{Z}_t^i)\Big) e^{ct}f'(r_t^i)dt. \end{split}$$

Now, we need to use the following Lemma, proven in Appendix A.6, to have a more tractable expression

Lemma 3.7. We have the upper bound

$$\begin{split} 2\epsilon \left(1 + \frac{1}{N}\right) \sigma_X^2 \varphi_{rc} \Big(|X_t^{i,N} - \bar{X}_t^i|\Big)^2 \text{sign}(X_t^{i,N} - \bar{X}_t^i) \left(\partial_X \tilde{H}(Z_t^{i,N}) - \partial_X \tilde{H}(\bar{Z}_t^i)\right) \\ & \leq \left(\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2}\right) \sigma_X^2 \varphi_{rc} \Big(|X_t^{i,N} - \bar{X}_t^i|\Big)^2 r_t^i G_t^i. \end{split}$$

Eventually, by denoting the terms in $dB_t^{i,rc,X}$, $dB_t^{i,sc,X}$, $dB_t^{i,C}$, ... as the local martingale dM_t^i , we obtain

$$\begin{split} d(e^{ct}f(r_t^i)G_t^i) &\leq G_t^ice^{ct}f(r_t^i)dt + e^{ct}G_t^if'(r_t^i)\left(A_t^X + \delta A_t^C\right)dt \\ &+ e^{ct}G_t^i\frac{1}{2}f''(r_t^i)\left(2\sigma_X\varphi_{\rm rc}\left(|X_t^{i,N} - \bar{X}_t^i|\right)\right)^2dt \\ &+ e^{ct}f(r_t^i)\left(\left(\mathcal{L}_{\bar{\mu}_t}\tilde{H}(\bar{Z}_t^i) + \mathcal{L}^N\tilde{H}(Z_t^{i,N})\right)dt + \frac{1}{N}\sum_{j=1}^N\left(\mathcal{L}_{\bar{\mu}_t}\tilde{H}(\bar{Z}_t^j) + \mathcal{L}^N\tilde{H}(Z_t^{j,N})\right)dt\right) \\ &+ (\epsilon\mathcal{C}_{f,1} + \mathcal{C}_{f,2})\,\sigma_X^2\varphi_{\rm rc}\Big(|X_t^{i,N} - \bar{X}_t^i|\Big)^2r_t^iG_t^ie^{ct}f'(r_t^i)dt + dM_t^i. \end{split}$$

We use (2.14) to bound $\mathcal{L}_{\bar{\mu}_t}\tilde{H}(\bar{Z}_t^i)$ and (2.17) to bound $\mathcal{L}^N\tilde{H}(Z_t^{i,N})$. The interaction terms

in A_t^X and A_t^C are decomposed and we define $I_t^{1,i}$ as follows

$$I_{t}^{1,i} = G_{t}^{i} f'(r_{t}^{i}) \left(\left| \frac{1}{N} \sum_{j=1}^{N} K_{X}(\bar{Z}_{t}^{i} - \bar{Z}_{t}^{j}) - K_{X} * \bar{\mu}_{t}(\bar{Z}_{t}^{i}) \right| \right) + \delta G_{t}^{i} f'(r_{t}^{i}) \left(\left| \frac{1}{N} \sum_{j=1}^{N} K_{C}(\bar{Z}_{t}^{i} - \bar{Z}_{t}^{j}) - K_{C} * \bar{\mu}_{t}(\bar{Z}_{t}^{i}) \right| \right).$$

The second part of the decomposition is grouped in $I_t^{2,i}$, with compensating terms that appear with the use of (2.14) and (2.17), to control the sum

$$\begin{split} I_{t}^{2,i} = & G_{t}^{i} f'(r_{t}^{i}) \left(\frac{L_{X}}{N} \left(\sum_{j=1}^{N} |X_{t}^{j,N} - \bar{X}_{t}^{j}| + |C_{t}^{j,N} - \bar{C}_{t}^{j}| \right) \right) \\ & + \delta G_{t}^{i} f'(r_{t}^{i}) \left(\frac{L_{C}}{N} \left(\sum_{j=1}^{N} |X_{t}^{j,N} - \bar{X}_{t}^{j}| + |C_{t}^{j,N} - \bar{C}_{t}^{j}| \right) \right) \\ & - c f(r_{t}^{i}) G_{t}^{i} - \epsilon f(r_{t}^{i}) \left[\frac{\lambda}{16} H(\bar{Z}_{t}^{i}) \exp\left(a \sqrt{H(\bar{Z}_{t}^{i})}\right) + \frac{\lambda}{16} H(Z_{t}^{i,N}) \exp\left(a \sqrt{H(Z_{t}^{i,N})}\right) \right] \\ & - \epsilon f(r_{t}^{i}) \frac{\lambda}{16N} \left[\sum_{j=1}^{N} H(\bar{Z}_{t}^{j}) \exp\left(a \sqrt{H(\bar{Z}_{t}^{j})}\right) + \sum_{j=1}^{N} H(Z_{t}^{j,N}) \exp\left(a \sqrt{H(Z_{t}^{j,N})}\right) \right]. \end{split}$$

We gather the expectations terms, obtained with (2.17), in $I_t^{3,i}$, and we keep a fraction of the Lyapunov function to control it

$$I_{t}^{3,i} = \epsilon f(r_{t}^{i}) \left((\alpha_{X} L_{X} + \beta_{X} L_{C}) \left(\frac{\sum_{j=1}^{N} |X_{t}^{j,N}|}{N} \right)^{2} \exp\left(a \sqrt{H(Z_{t}^{i,N})} \right) + (\alpha_{C} L_{X} + \beta_{C} L_{C}) \left(\frac{\sum_{j=1}^{N} |C_{t}^{j,N}|}{N} \right)^{2} \exp\left(a \sqrt{H(Z_{t}^{i,N})} \right) - \frac{\lambda}{16N} \sum_{j=1}^{N} H(Z_{t}^{j,N}) \exp\left(a \sqrt{H(Z_{t}^{j,N})} \right) \right).$$

Finally, we define \tilde{K}^i_t with the leftovers. It will, in particular, give the constraints on f which explain its choice.

$$\begin{split} \tilde{K}_{t}^{i} = & G_{t}^{i} \bigg[2cf(r_{t}^{i}) + \frac{1}{2}f''(r_{t}^{i}) \left(2\sigma_{X}^{2}\varphi_{\text{rc}} \Big(|X_{t}^{i,N} - \bar{X}_{t}^{i}| \Big)^{2} \right) \\ & + f'(r_{t}^{i}) \Big((1 + \gamma\delta + L_{X} + \delta L_{C}) |X_{t}^{i,N} - \bar{X}_{t}^{i}| - |(X_{t}^{i,N})^{3} - (\bar{X}_{t}^{i})^{3}| \\ & + (1 + L_{X} + \delta L_{C} - \delta) |C_{t}^{i,N} - \bar{C}_{t}^{i}| + (\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2}) \, \sigma_{X}^{2} \varphi_{\text{rc}} \Big(|X_{t}^{i,N} - \bar{X}_{t}^{i}| \Big)^{2} r_{t}^{i} \Big) \bigg] \\ & + \epsilon f(r_{t}^{i}) \left(4\tilde{B} - \frac{\lambda}{16} \tilde{H}(\bar{Z}_{t}^{i}) - \frac{\lambda}{16} \tilde{H}(Z_{t}^{i,N}) - \frac{\lambda}{16N} \sum_{j=1}^{N} \tilde{H}(\bar{Z}_{t}^{j}) - \frac{\lambda}{16N} \sum_{j=1}^{N} \tilde{H}(Z_{t}^{j,N}) \right). \end{split}$$

3.3 Controls of $I_t^{1,i}$, $I_t^{2,i}$ and $I_t^{3,i}$

Proof of Lemma 3.6. Since we assume

$$\frac{4}{\gamma}(\alpha_X L_X + \beta_X L_C) \le \frac{\lambda}{32}$$
 and $4(\alpha_C L_X + \beta_C L_C) \le \frac{\lambda}{32}$

and since

$$H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{i,N})}\right) \leq H(Z_t^{i,N}) \exp\left(a\sqrt{H(Z_t^{i,N})}\right) + H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{j,N})}\right)$$

we obtain

$$(\alpha_X L_X + \beta_X L_C) \left(\frac{\sum_{j=1}^N |X_t^{j,N}|}{N}\right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})}\right)$$

$$+ (\alpha_C L_X + \beta_C L_C) \left(\frac{\sum_{j=1}^N |C_t^{j,N}|}{N}\right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})}\right)$$

$$- \frac{\lambda}{16N} \left(NH(Z_t^{i,N}) \exp\left(a\sqrt{H(Z_t^{i,N})}\right) + \sum_{j=1}^N H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right) \le 0.$$

Then, for each $i \leq N$, and for all t > 0, $I_t^{3,i} \leq 0$.

Proof of Lemma 3.5. We prove the non-positivity of $\frac{1}{N}\sum_{i=1}^{N}I_{t}^{2,i}$. First, since $f'\left(r_{t}^{i}\right)\leq1$, we have

$$\begin{split} &\frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{N} f'\left(r_{t}^{i}\right) G_{t}^{i} \sum_{j=1}^{N} \|Z_{t}^{j,N} - \bar{Z}_{t}^{j}\|_{1} \right) \\ &\leq & \frac{1}{N} \sum_{i=1}^{N} \|Z_{t}^{i,N} - \bar{Z}_{t}^{i}\|_{1} + \frac{2\epsilon}{N^{2}} \sum_{i,j=1}^{N} \|Z_{t}^{i,N} - \bar{Z}_{t}^{i}\|_{1} \left(\tilde{H}\left(\bar{Z}_{t}^{j}\right) + \tilde{H}(Z_{t}^{j,N}) \right), \end{split}$$

and, using Lemma 2.7 (i)

$$\frac{1}{N} \sum_{i=1}^{N} \|Z_t^{i,N} - \bar{Z}_t^i\|_1 \le \frac{\mathcal{C}_1}{N} \sum_{i=1}^{N} f(r_t^i) G_t^i,$$

and with Lemma 2.7 (iii)

$$\begin{split} \sum_{i,j=1}^{N} \|Z_t^{i,N} - \bar{Z}_t^i\|_1 \left(\tilde{H}\left(\bar{Z}_t^j\right) + \tilde{H}(Z_t^{j,N}) \right) &\leq \mathcal{C}_z \sum_{i,j=1}^{N} f(r_t^i) \left(\tilde{H}\left(\bar{Z}_t^j\right) + \tilde{H}(Z_t^{j,N}) \right) \\ &+ \epsilon \mathcal{C}_z \sum_{i,j=1}^{N} f(r_t^i) \left(\sqrt{H(Z_t^{i,N})} + \sqrt{H(\bar{Z}_t^i)} \right) \left(\tilde{H}\left(\bar{Z}_t^j\right) + \tilde{H}(Z_t^{j,N}) \right). \end{split}$$

Using (2.9) from Lemma 2.5, we obtain for the first sum

$$C_z \sum_{i,j=1}^N f(r_t^i) \left(\tilde{H} \left(\bar{Z}_t^j \right) + \tilde{H}(Z_t^{j,N}) \right)$$

$$\leq C_z \sum_{i,j=1}^N f(r_t^i) \left(H(\bar{Z}_t^j) \exp\left(a \sqrt{H(\bar{Z}_t^j)} \right) + H(Z_t^{j,N}) \exp\left(a \sqrt{H(Z_t^{j,N})} \right) \right).$$

With (2.10) from the same Lemma, we obtain for the second sum

$$\begin{split} \epsilon \mathcal{C}_{z} \sum_{i,j=1}^{N} f(r_{t}^{i}) \left(\sqrt{H(Z_{t}^{i,N})} + \sqrt{H(\bar{Z}_{t}^{i})} \right) \left(\tilde{H} \left(\bar{Z}_{t}^{j} \right) + \tilde{H}(Z_{t}^{j,N}) \right) \\ \leq \epsilon \mathcal{C}_{z} \frac{2}{a} \sum_{i,j=1}^{N} f(r_{t}^{i}) \left(\sqrt{H(Z_{t}^{i,N})} + \sqrt{H(\bar{Z}_{t}^{i})} \right) \\ \times \left(\sqrt{H \left(\bar{Z}_{t}^{j} \right)} \exp \left(a \sqrt{H \left(\bar{Z}_{t}^{j} \right)} \right) + \sqrt{H \left(Z_{t}^{j,N} \right)} \exp \left(a \sqrt{H \left(Z_{t}^{j,N} \right)} \right) \right). \end{split}$$

Since for all $(y_1, y_2, y_3, y_4) \in (\mathbb{R}^+)^4$, we have

$$(y_1 + y_2)(y_3e^{ay_3} + y_4e^{ay_4}) \le 2(y_1^2e^{ay_1} + y_2^2e^{ay_2} + y_3^2e^{ay_3} + y_4^2e^{ay_4}),$$

we obtain for this last sum

$$\begin{split} \frac{2\epsilon \mathcal{C}_z}{a} \sum_{i,j=1}^N f(r_t^i) \left(\sqrt{H(Z_t^{i,N})} + \sqrt{H(\bar{Z}_t^i)} \right) \\ & \times \left(\sqrt{H\left(\bar{Z}_t^j\right)} \exp\left(a\sqrt{H\left(\bar{Z}_t^j\right)}\right) + \sqrt{H\left(Z_t^{j,N}\right)} \exp\left(a\sqrt{H\left(Z_t^{j,N}\right)}\right) \right) \\ \leq & \frac{4\epsilon \mathcal{C}_z}{a} N \sum_{i=1}^N f(r_t^i) \left(H(\bar{Z}_t^i) \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right) + H(Z_t^{i,N}) \exp\left(a\sqrt{H(Z_t^{i,N})}\right) \right) \\ & + \frac{4\epsilon \mathcal{C}_z}{a} \sum_{i,j=1}^N f(r_t^i) \left(H(\bar{Z}_t^j) \exp\left(a\sqrt{H(\bar{Z}_t^j)}\right) + H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{j,N})}\right) \right). \end{split}$$

Then, by reconsidering the first expression

$$\begin{split} &\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{N}f'\left(r_{t}^{i}\right)G_{t}^{i}\sum_{j=1}^{N}\|Z_{t}^{j,N}-\bar{Z}_{t}^{j}\|_{1}\right)\\ &\leq &\frac{\mathcal{C}_{1}}{N}\sum_{i=1}^{N}f(r_{t}^{i})G_{t}^{i}\\ &+\frac{2\epsilon}{N^{2}}\mathcal{C}_{z}\sum_{i,j=1}^{N}f(r_{t}^{i})\left(H(\bar{Z}_{t}^{j})\exp\left(a\sqrt{H(\bar{Z}_{t}^{j})}\right)+H(Z_{t}^{j,N})\exp\left(a\sqrt{H(Z_{t}^{j,N})}\right)\right)\\ &+\frac{2\epsilon}{N^{2}}\frac{4\epsilon\mathcal{C}_{z}}{a}N\sum_{i=1}^{N}f(r_{t}^{i})\left(H(\bar{Z}_{t}^{i})\exp\left(a\sqrt{H(\bar{Z}_{t}^{i})}\right)+H(Z_{t}^{i,N})\exp\left(a\sqrt{H(Z_{t}^{i,N})}\right)\right)\\ &+\frac{2\epsilon}{N^{2}}\frac{4\epsilon\mathcal{C}_{z}}{a}\sum_{i,j=1}^{N}f(r_{t}^{i})\left(H(\bar{Z}_{t}^{j})\exp\left(a\sqrt{H(\bar{Z}_{t}^{j})}\right)+H(Z_{t}^{j,N})\exp\left(a\sqrt{H(Z_{t}^{i,N})}\right)\right) \end{split}$$

This way, by (2.21) since

$$L_X \mathcal{C}_1 \leq \frac{c}{2}, \quad 2\mathcal{C}_z L_X \leq \frac{\lambda}{64} \quad \text{and} \quad L_X \epsilon \frac{8\mathcal{C}_z}{a} \leq \frac{\lambda}{64},$$

we get

$$\begin{split} &\frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{N} f'\left(r_{t}^{i}\right) G_{t}^{i} \sum_{j=1}^{N} \|Z_{t}^{j,N} - \bar{Z}_{t}^{j}\|_{1} \right) \\ &\leq \frac{1}{N} \frac{c}{2L_{X}} \sum_{i=1}^{N} f(r_{t}^{i}) G_{t}^{i} \\ &\quad + \frac{\epsilon}{N^{2}} \frac{\lambda}{64L_{X}} \sum_{i,j=1}^{N} f(r_{t}^{i}) \left(H(\bar{Z}_{t}^{j}) \exp\left(a\sqrt{H(\bar{Z}_{t}^{j})}\right) + H(Z_{t}^{j,N}) \exp\left(a\sqrt{H(Z_{t}^{j,N})}\right) \right) \\ &\quad + \frac{\epsilon}{N} \frac{\lambda}{64L_{X}} \sum_{i=1}^{N} f(r_{t}^{i}) \left(H(\bar{Z}_{t}^{i}) \exp\left(a\sqrt{H(\bar{Z}_{t}^{i})}\right) + H(Z_{t}^{i,N}) \exp\left(a\sqrt{H(Z_{t}^{i,N})}\right) \right) \\ &\quad + \frac{\epsilon}{N^{2}} \frac{\lambda}{64L_{X}} \sum_{i=1}^{N} f(r_{t}^{i}) \left(H(\bar{Z}_{t}^{j}) \exp\left(a\sqrt{H(\bar{Z}_{t}^{j})}\right) + H(Z_{t}^{j,N}) \exp\left(a\sqrt{H(Z_{t}^{j,N})}\right) \right), \end{split}$$

and we finally obtain "half" the result

$$\begin{split} &\frac{1}{N} \sum_{i=1}^{N} G_{t}^{i} f'(r_{t}^{i}) \left(\frac{L_{X}}{N} \left(\sum_{j=1}^{N} \|Z_{t}^{j,N} - \bar{Z}_{t}^{j}\|_{1} \right) \right) - \frac{c}{2} \frac{1}{N} \sum_{i=1}^{N} f(r_{t}^{i}) G_{t}^{i} \\ &- \frac{\epsilon}{2} \frac{\lambda}{16N} \sum_{i=1}^{N} f(r_{t}^{i}) \left[H(\bar{Z}_{t}^{i}) \exp\left(a\sqrt{H(\bar{Z}_{t}^{i})}\right) + H(Z_{t}^{i,N}) \exp\left(a\sqrt{H(Z_{t}^{i,N})}\right) \right] \\ &- \frac{\epsilon}{2} \frac{\lambda}{16N^{2}} \sum_{i=1}^{N} f(r_{t}^{i}) \left[\sum_{j=1}^{N} H(\bar{Z}_{t}^{j}) \exp\left(a\sqrt{H(\bar{Z}_{t}^{j})}\right) + \sum_{j=1}^{N} H(Z_{t}^{j,N}) \exp\left(a\sqrt{H(Z_{t}^{j,N})}\right) \right] \leq 0. \end{split}$$

Likewise, by (2.21), since

$$\delta L_C C_1 \le \frac{c}{2}, \quad 2C_z \delta L_C \le \frac{\lambda}{64} \quad \text{and} \quad \delta L_C \epsilon \frac{8C_z}{a} \le \frac{\lambda}{64},$$

we obtain the second "half"

$$\frac{1}{N} \sum_{i=1}^{N} \delta G_{t}^{i} f'(r_{t}^{i}) \left(\frac{L_{C}}{N} \left(\sum_{j=1}^{N} \| Z_{t}^{j,N} - \bar{Z}_{t}^{j} \|_{1} \right) \right) - \frac{c}{2} \frac{1}{N} \sum_{i=1}^{N} f(r_{t}^{i}) G_{t}^{i}
- \frac{\epsilon}{2} \frac{\lambda}{16N} \sum_{i=1}^{N} f(r_{t}^{i}) \left[H(\bar{Z}_{t}^{i}) \exp\left(a\sqrt{H(\bar{Z}_{t}^{i})}\right) + H(Z_{t}^{i,N}) \exp\left(a\sqrt{H(Z_{t}^{i,N})}\right) \right]
- \frac{\epsilon}{2} \frac{\lambda}{16N^{2}} \sum_{i=1}^{N} f(r_{t}^{i}) \left[\sum_{j=1}^{N} H(\bar{Z}_{t}^{j}) \exp\left(a\sqrt{H(\bar{Z}_{t}^{j})}\right) + \sum_{j=1}^{N} H(Z_{t}^{j,N}) \exp\left(a\sqrt{H(Z_{t}^{j,N})}\right) \right] \le 0.$$

Eventually, we have proved $\sum_{i=1}^N I_t^{2,i} \leq 0.$

Proof of Lemma 3.4. Since $f'(r) \leq 1$, we have by Cauchy-Schwarz inequality

$$\mathbb{E}\left(G_{t}^{i}f'(r_{t}^{i})\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_{X}(\bar{Z}_{t}^{i}-\bar{Z}_{t}^{j})-K_{X}*\bar{\mu}_{t}(\bar{Z}_{t}^{i})\right|\right)\right)$$

$$\leq \mathbb{E}\left(|G_{t}^{i}|^{2}\right)^{1/2}\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_{X}(\bar{Z}_{t}^{i}-\bar{Z}_{t}^{j})-K_{X}*\bar{\mu}_{t}(\bar{Z}_{t}^{i})\right|^{2}\right)^{1/2}.$$

By Lemma 3.2, we have for each $i \leq N$, for all $t \geq 0$, $\mathbb{E}[(G_t^i)^2] \leq C_{G,2}$.

Moreover, we notice that the $(\bar{Z}_t^j)_j$ are i.i.d with law $\bar{\mu}_t$. Let's denote \bar{Z}_t a generic random variable of law $\bar{\mu}_t$ independent of \bar{Z}_t^i . The calculus of the right term of the product has already been done in Subsection 1.4, and we have (1.10)

$$\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_{X}(\bar{Z}_{t}^{i}-\bar{Z}_{t}^{j})-K_{X}*\bar{\mu}_{t}(\bar{Z}_{t}^{i})\right|^{2}\right)\leq\frac{8L_{X}^{2}}{N}\mathbb{E}(\|\bar{Z}_{t}\|_{1}^{2}).$$

A similar calculation yields

$$\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_{C}(\bar{Z}_{t}^{i}-\bar{Z}_{t}^{j})-K_{C}*\bar{\mu}_{t}(\bar{Z}_{t}^{i})\right|^{2}\right)\leq\frac{8L_{C}^{2}}{N}\mathbb{E}(\|\bar{Z}_{t}\|_{1}^{2}).$$

By Lemma 2.4, $\mathbb{E}\left(|\bar{X}_t|^2+|\bar{C}_t|^2\right)\leq \mathcal{C}_{init,2}$. In particular,

$$\mathbb{E}\left(\|\bar{Z}_t\|_1^2\right) = \mathbb{E}\left(\left[|\bar{X}_t| + |\bar{C}_t|\right]^2\right) \le 2\mathbb{E}\left(|\bar{X}_t|^2 + |\bar{C}_t|^2\right) \le 2\mathcal{C}_{init,2}.$$

Thus

$$\mathbb{E}\left(G_t^i f'(r_t^i) \left(\left| \frac{1}{N} \sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i) \right| \right) \right) \le L_X \mathcal{C}_{G,2}^{1/2} \sqrt{2\mathcal{C}_{init,2}} \sqrt{\frac{8}{N}},$$

and likewise

$$\mathbb{E}\left(G_t^i f'(r_t^i) \left(\left| \frac{1}{N} \sum_{j=1}^N K_C(\bar{Z}_t^i - \bar{Z}_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i) \right| \right) \right) \le L_C \mathcal{C}_{G,2}^{1/2} \sqrt{2\mathcal{C}_{init,2}} \sqrt{\frac{8}{N}}.$$

3.4 Contraction in various regions of space

The goal of this section is to prove Lemma 3.3, i.e show that for each $i \leq N$, for all t > 0, we have the following control

$$\mathbb{E}\tilde{K}_t^i \le \xi \left(2 + \delta \gamma + L_X + \delta L_C - L_C - \frac{1 + L_X}{\delta} \right) \mathbb{E}G_t^i.$$

Recall

$$\begin{split} \tilde{K}_{t}^{i} = & G_{t}^{i} \bigg[2cf(r_{t}^{i}) + \frac{1}{2}f''(r_{t}^{i}) \left(2\sigma_{X}^{2}\varphi_{\text{rc}} \Big(|X_{t}^{i,N} - \bar{X}_{t}^{i}| \Big)^{2} \right) \\ & + f'(r_{t}^{i}) \Big((1 + \gamma\delta + L_{X} + \delta L_{C}) |X_{t}^{i,N} - \bar{X}_{t}^{i}| - |(X_{t}^{i,N})^{3} - (\bar{X}_{t}^{i})^{3}| \\ & + (1 + L_{X} + \delta L_{C} - \delta) |C_{t}^{i,N} - \bar{C}_{t}^{i}| + (\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2}) \, \sigma_{X}^{2} \varphi_{\text{rc}} \Big(|X_{t}^{i,N} - \bar{X}_{t}^{i}| \Big)^{2} r_{t}^{i} \Big) \bigg] \\ & + \epsilon f(r_{t}^{i}) \left(4\tilde{B} - \frac{\lambda}{16} \tilde{H}(\bar{Z}_{t}^{i}) - \frac{\lambda}{16} \tilde{H}(Z_{t}^{i,N}) - \frac{\lambda}{16N} \sum_{j=1}^{N} \tilde{H}(\bar{Z}_{t}^{j}) - \frac{\lambda}{16N} \sum_{j=1}^{N} \tilde{H}(Z_{t}^{j,N}) \right), \end{split}$$

which is a quantity that contains every term we have not yet dealt with. To prove Lemma 3.3, we divide for each $i \in \{1, ..., N\}$ the space into three regions

$$\begin{split} \operatorname{Reg}_1^i &= \left\{ (\bar{Z}_t^i, Z_t^{i,N}) \text{ s.t. } |\bar{X}_t^i - X_t^{i,N}| \geq \xi \text{ and } r_t^i \leq R \right\}, \\ \operatorname{Reg}_2^i &= \left\{ (\bar{Z}_t^i, Z_t^{i,N}) \text{ s.t. } |\bar{X}_t^i - X_t^{i,N}| < \xi \text{ and } r_t^i \leq R_1 \right\}, \\ \operatorname{Reg}_3^i &= \left\{ (\bar{Z}_t^i, Z_t^{i,N}) \text{ s.t. } r_t^i > R \right\}, \end{split}$$

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where R was given in Lemma 2.1, and consider

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\tilde{K}_{t}^{i} = \frac{1}{N}\sum_{i=1}^{N}\left(\mathbb{E}\left(\tilde{K}_{t}^{i}\mathbb{1}_{\mathbf{Reg}_{1}^{i}}\right) + \mathbb{E}\left(\tilde{K}_{t}^{i}\mathbb{1}_{\mathbf{Reg}_{2}^{i}}\right) + \mathbb{E}\left(\tilde{K}_{t}^{i}\mathbb{1}_{\mathbf{Reg}_{3}^{i}}\right)\right).$$

3.4.1 Region 1: $\xi \leq |X_t^{i,N} - \bar{X}_t^i|$ and $r_t^i \leq R$.

In this region of space, since $\varphi_{\rm rc}(|X_t^{i,N}-\bar{X}_t^i|)=1$, we have

$$\begin{split} \tilde{K}_{t}^{i} \mathbb{1}_{\text{Reg}_{1}^{i}} = & \mathbb{1}_{\text{Reg}_{1}^{i}} \left(G_{t}^{i} \Big[2cf(r_{t}^{i}) + 2\sigma_{X}^{2}f''(r_{t}^{i}) + f'(r_{t}^{i}) \left(\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2} \right) \sigma_{X}^{2}r_{t}^{i} \right. \\ & + f'(r_{t}^{i})(1 + \gamma \delta + L_{X} + \delta L_{C}) |X_{t}^{i,N} - \bar{X}_{t}^{i}| \Big] \\ & - G_{t}^{i}f'(r_{t}^{i})(\delta - 1 - L_{X} - \delta L_{C}) |C_{t}^{i,N} - \bar{C}_{t}^{i}| - G_{t}^{i}f'(r_{t}^{i}) |(X_{t}^{i,N})^{3} - (\bar{X}_{t}^{i})^{3}| \\ & + \epsilon f(r_{t}^{i})4\tilde{B} \\ & - \epsilon f(r_{t}^{i}) \left(\frac{\lambda}{16}\tilde{H}(\bar{Z}_{t}^{i}) + \frac{\lambda}{16}\tilde{H}(Z_{t}^{i,N}) + \frac{\lambda}{16N} \sum_{j=1}^{N} \tilde{H}(\bar{Z}_{t}^{j}) + \frac{\lambda}{16N} \sum_{j=1}^{N} \tilde{H}(Z_{t}^{j,N}) \right) \right), \end{split}$$

and since $\tilde{H}(z) \geq 0$, $|X_t^{i,N} - \bar{X}_t^i| \leq r_t^i$, $\delta > \frac{1+L_X}{1-L_C}$ (by the choice given in Subsection 2.4) and $1 \leq G_t^i$ we have

$$\begin{split} \tilde{K}_t^i \mathbb{1}_{\mathbf{Reg}_1^i} \leq & \mathbb{1}_{\mathbf{Reg}_1^i} G_t^i \left[(2c + 4\epsilon \tilde{B}) f(r_t^i) + 2\sigma_X^2 f''(r_t^i) \right. \\ & \left. + f'(r_t^i) \left(1 + \delta \gamma + L_X + \delta L_C + (\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2}) \, \sigma_X^2 \right) r_t^i \right]. \end{split}$$

Using the definition f given in (2.20) we get

$$\begin{split} 2\sigma_{X}^{2}f''(r_{t}^{i}) + f'(r_{t}^{i}) \left(1 + \delta\gamma + L_{X} + L_{C} + (\epsilon\mathcal{C}_{f,1} + \mathcal{C}_{f,2}) \, \sigma_{X}^{2}\right) r_{t}^{i} \\ &= 2\sigma_{X}^{2}\phi'(r_{t}^{i})g(r_{t}^{i}) + 2\sigma_{X}^{2}\phi(r_{t}^{i})g'(r_{t}^{i}) \\ &+ \phi(r_{t}^{i})g(r_{t}^{i}) \left(1 + \delta\gamma + L_{X} + \delta L_{C} + (\epsilon\mathcal{C}_{f,1} + \mathcal{C}_{f,2}) \, \sigma_{X}^{2}\right) r_{t}^{i} \\ &= 2\sigma_{X}^{2}\phi(r_{t}^{i})g'(r_{t}^{i}) = -(2c + 4\epsilon\tilde{B})\Phi(r_{t}^{i}). \end{split}$$

Thus

$$(2c + 4\epsilon \tilde{B})f(r_t^i) + 2\sigma_X^2 f''(r_t^i) + f'(r_t^i) \left(1 + \delta \gamma + L_X + \delta L_C + (\epsilon C_{f,1} + C_{f,2}) \sigma_X^2\right) r_t^i$$

$$= (2c + 4\epsilon \tilde{B})f(r_t^i) - (2c + 4\epsilon \tilde{B})\Phi(r_t^i)$$

$$\leq 0.$$
(3.12)

Eventually, in this region of space

$$\tilde{K}_t^i \mathbb{1}_{\operatorname{Reg}_1^i} \leq 0.$$

3.4.2 Region 2: $|X_t^{i,N} - \bar{X}_t^i| < \xi$ and $r_t^i \le R$.

In this region, we can write $ilde{K}^i_t$ as

$$\begin{split} \tilde{K}_{t}^{i} \mathbb{1}_{\text{Reg}_{2}^{i}} &= \mathbb{1}_{\text{Reg}_{2}^{i}} G_{t}^{i} \Big[2cf(r_{t}^{i}) + \varphi_{\text{rc}} \Big(|X_{t}^{i,N} - \bar{X}_{t}^{i}| \Big)^{2} \, \Big[2\sigma_{X}^{2} f''(r_{t}^{i}) + (\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2}) \, \sigma_{X}^{2} r_{t}^{i} f'(r_{t}^{i}) \Big] \\ &+ f'(r_{t}^{i}) \, \Big((1 + \gamma \delta + L_{X} + \delta L_{C}) |X_{t}^{i,N} - \bar{X}_{t}^{i}| - (\delta - 1 - L_{X} - \delta L_{C}) |C_{t}^{i,N} - \bar{C}_{t}^{i}| \Big) \Big] \\ &- \mathbb{1}_{\text{Reg}_{2}^{i}} G_{t}^{i} f'(r_{t}^{i}) |(X_{t}^{i,N})^{3} - (\bar{X}_{t}^{i})^{3}| \\ &+ \mathbb{1}_{\text{Reg}_{2}^{i}} \epsilon f(r_{t}^{i}) 4\tilde{B} \\ &- \epsilon f(r_{t}^{i}) \mathbb{1}_{\text{Reg}_{2}^{i}} \left(\frac{\lambda}{16} \tilde{H}(\bar{Z}_{t}^{i}) + \frac{\lambda}{16} \tilde{H}(Z_{t}^{i,N}) + \frac{\lambda}{16N} \sum_{j=1}^{N} \tilde{H}(\bar{Z}_{t}^{j}) + \frac{\lambda}{16N} \sum_{j=1}^{N} \tilde{H}(Z_{t}^{j,N}) \right). \end{split}$$

Since $r_t^i = |X_t^{i,N} - \bar{X}_t^i| + \delta |C_t^{i,N} - \bar{C}_t^i|$ and $|X_t^{i,N} - \bar{X}_t^i| < \xi$, we have $|C_t^{i,N} - \bar{C}_t^i| \ge (r_t^i - \xi)/\delta$. Since $\delta > \frac{1 + L_X}{1 - L_C}$, we obtain

$$\begin{split} \tilde{K}_{t}^{i} \mathbb{1}_{\text{Reg}_{2}^{i}} \leq & \mathbb{1}_{\text{Reg}_{2}^{i}} G_{t}^{i} \Big[2cf(r_{t}^{i}) + \varphi_{\text{rc}} \Big(|X_{t}^{i,N} - \bar{X}_{t}^{i}| \Big)^{2} \left[2\sigma_{X}^{2} f''(r_{t}^{i}) + (\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2}) \, \sigma_{X}^{2} r_{t}^{i} f'(r_{t}^{i}) \right] \\ & + f'(r_{t}^{i}) \left((1 + \gamma \delta + L_{X} + \delta L_{C}) \xi - (\delta - \delta L_{C} - 1 - L_{X}) \frac{r_{t}^{i} - \xi}{\delta} \right) \Big] \\ & + \epsilon f(r_{t}^{i}) \mathbb{1}_{\text{Reg}_{2}^{i}} 4\tilde{B} \\ \leq & \varphi_{\text{rc}} \Big(|X_{t}^{i,N} - \bar{X}_{t}^{i}| \Big)^{2} G_{t}^{i} \mathbb{1}_{\text{Reg}_{2}^{i}} \Big[2\sigma_{X}^{2} f''(r_{t}^{i}) + (\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2}) \, \sigma_{X}^{2} r_{t}^{i} f'(r_{t}^{i}) \Big] \\ & + \mathbb{1}_{\text{Reg}_{2}^{i}} G_{t}^{i} f'(r_{t}^{i}) \xi \left[1 + \gamma \delta + L_{X} + \delta L_{C} + 1 - L_{C} - \frac{1 + L_{X}}{\delta} \right] \\ & + \mathbb{1}_{\text{Reg}_{2}^{i}} G_{t}^{i} \left((2c + 4\epsilon \tilde{B}) f(r_{t}^{i}) - r_{t}^{i} f'(r_{t}^{i}) \left(1 - L_{C} - \frac{1 + L_{X}}{\delta} \right) \right). \end{split}$$

By (3.12),

$$2\sigma_X^2 f''(r_t^i) + (\epsilon C_{f,1} + C_{f,2}) \sigma_X^2 r_t^i f'(r_t^i) = -(2c + 4\epsilon \tilde{B}) \Phi(r_t^i) - f'(r_t^i) r_t^i (1 + \delta \gamma + L_X + L_C)$$

$$< 0.$$

and by Lemma 2.6

$$2c + 4\epsilon \tilde{B} \le \left(1 - L_C - \frac{1 + L_X}{\delta}\right) \min_{r \in (0, R]} \frac{f'(r)r}{f(r)},$$

we obtain

$$\tilde{K}_t^i \mathbb{1}_{\mathsf{Reg}_2^i} \leq \mathbb{1}_{\mathsf{Reg}_2^i} G_t^i f'(r_t^i) \xi \left[1 + \gamma \delta + L_X + \delta L_C + 1 - L_C - \frac{1 + L_X}{\delta} \right].$$

Finally, since $f'(r) \leq 1$,

$$\mathbb{E} \tilde{K}_t^i \mathbb{1}_{\mathsf{Reg}_2^i} \leq \xi \left(2 + \delta \gamma + L_X + \delta L_C - L_C - \frac{1 + L_X}{\delta} \right) \mathbb{E} G_t^i.$$

3.4.3 Region 3: $r_t^i \ge R$.

In this region of space f' = f'' = 0 and f is constant, and we therefore have

$$\begin{split} \tilde{K}_t^i \mathbb{1}_{\mbox{Reg}_3^i} = & f(r_t^i) \mathbb{1}_{\mbox{Reg}_3^i} \left[2cG_t^i + 4\epsilon \tilde{B} \right. \\ & \left. - \frac{\lambda \epsilon}{16} \left(\tilde{H}(\bar{Z}_t^i) + \tilde{H}(Z_t^{i,N}) + \frac{1}{N} \sum_{j=1}^N \tilde{H}(\bar{Z}_t^j) + \frac{1}{N} \sum_{j=1}^N \tilde{H}(Z_t^{j,N}) \right) \right]. \end{split}$$

Since $G_t^i = 1 + \epsilon \tilde{H}(\bar{Z}_t^i) + \epsilon \tilde{H}(Z_t^{i,N}) + \frac{\epsilon}{N} \sum_{j=1}^N \tilde{H}(Z_t^{j,N}) + \frac{\epsilon}{N} \sum_{j=1}^N \tilde{H}(\bar{Z}_t^j)$ by definition (2.26), we can write

$$\begin{split} \tilde{K}_t^i \mathbb{1}_{\mathsf{Reg}_3^i} = & f(r_t^i) \mathbb{1}_{\mathsf{Reg}_3^i} \left[2c + 4\epsilon \tilde{B} \right. \\ & \left. + \epsilon \left(2c - \frac{\lambda}{16} \right) \left(\tilde{H}(\bar{Z}_t^i) + \tilde{H}(Z_t^{i,N}) + \frac{1}{N} \sum_{j=1}^N \tilde{H}(\bar{Z}_t^j) + \frac{1}{N} \sum_{j=1}^N \tilde{H}(Z_t^{j,N}) \right) \right]. \end{split}$$

Since $c \le \lambda/32$ by the choice given in Subsection 2.4, we obtain

$$\tilde{K}_t^i \mathbb{1}_{\mathsf{Reg}_3^i} \leq f(r_t^i) \mathbb{1}_{\mathsf{Reg}_3^i} \left[2c + 4\epsilon \tilde{B} - \epsilon \left(\frac{\lambda}{16} - 2c \right) \left(H(\bar{Z}_t^i) + H(Z_t^{i,N}) \right) \right].$$

We have chosen R such that, for z, z' satisfying $r \ge R$, we have $H(z) + H(z') \ge 80 \frac{\tilde{B}}{\lambda}$ by Lemma 2.1 (iv). Therefore

$$\begin{split} \tilde{K}_t^i \mathbb{1}_{\mathbf{Reg}_3^i} \leq & f(r_t^i) \mathbb{1}_{\mathbf{Reg}_3^i} \left(2c + 4\epsilon \tilde{B} - \epsilon \left(\frac{\lambda}{16} - 2c \right) 80 \frac{\tilde{B}}{\lambda} \right) \\ = & f(r_t^i) \mathbb{1}_{\mathbf{Reg}_3^i} \left(2c \left(1 + 80 \frac{\epsilon \tilde{B}}{\lambda} \right) - \epsilon \tilde{B} \right) \end{split}$$

Lemma 2.6 and more specifically the inequality

$$c \leq \frac{1}{2} \frac{\epsilon \tilde{B}}{1 + 80 \frac{\epsilon \tilde{B}}{\lambda}} = \frac{\lambda}{160} \frac{\frac{80 \epsilon \tilde{B}}{\lambda}}{1 + \frac{80 \epsilon \tilde{B}}{\lambda}}$$

yields the desired result: $\tilde{K}_t^i \mathbb{1}_{\text{Reg}_2^i} \leq 0$.

A Various technical lemmas

A.1 On Itô's formula for the L^1 norm

Let us here detail the calculations leading to the use of Itô's formula to derive the dynamics of the L^1 norm of the processes. At first glance it should not be possible, as the absolute value is not a twice continuously differentiable function. However, in our case, we consider a diffusion coefficient which is zero around the point of discontinuity of the function. The following lemma is based on the calculations done in Lemma 7 of [8], and relies on an approximation of the absolute value function and usual convergence lemmas. We here give a guite general result.

Lemma A.1. Let (X_t, X_t', C_t, C_t') be continuous processes and $F, G: \mathbb{R}^+ \times \mathbb{R}^4 \mapsto \mathbb{R}$ be two continuous functions. Assume furthermore that there is $R_G > 0$ such that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and that G(t, x, x', c, c') = 0 if $|x - x'| < R_G$ and $|x - x'| < R_G$ and

$$d(X_t - X_t') = F(t, X_t, X_t', C_t, C_t')dt + G(t, X_t, X_t', C_t, C_t')dB_t,$$

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where B is a Brownian motion. Then almost surely for all $t \geq 0$

$$d|X_{t} - X'_{t}| = sign(X_{t} - X'_{t})F(t, X_{t}, X'_{t}, C_{t}, C'_{t})dt + sign(X_{t} - X'_{t})G(t, X_{t}, X'_{t}, C_{t}, C'_{t})dB_{t},$$
(A.1)

where

$$sign(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Proof. By the standard Itô's formula for twice continuously differentiable functions, we have

$$d(X_t - X_t')^2 = 2(X_t - X_t')F(t, X_t, X_t', C_t, C_t')dt + 2(X_t - X_t')G(t, X_t, X_t', C_t, C_t')dB_t + G^2(t, X_t, X_t', C_t, C_t')dt.$$

Consider, for $\eta > 0$ (which in the end will go to 0), the function $\psi_{\eta}(r) = (r + \eta)^{1/2}$ which is smooth on $[0, \infty[$ and satisfies

$$\begin{split} \forall r > 0, \lim_{\eta \to 0} \psi_{\eta}(r) &= r^{1/2}, \lim_{\eta \to 0} 2 \psi_{\eta}'(r) = r^{-1/2}, \lim_{\eta \to 0} 4 \psi_{\eta}''(r) = -r^{-3/2} \\ \text{and thus } \lim_{\eta \to 0} 2 r \psi_{\eta}''(r) + \psi_{\eta}'(r) = 0 \text{ and } \forall r \in \mathbb{R}, \lim_{\eta \to 0} 2 r \psi_{\eta}'(r^2) = \mathrm{sign}(r). \end{split}$$

Then

$$\begin{split} d\psi_{\eta} \left(\left(X_{t} - X_{t}' \right)^{2} \right) = & 2 (X_{t} - X_{t}') \psi_{\eta}' \left(\left(X_{t} - X_{t}' \right)^{2} \right) F(t, X_{t}, X_{t}', C_{t}, C_{t}') dt \\ & + 2 (X_{t} - X_{t}') \psi_{\eta}' \left(\left(X_{t} - X_{t}' \right)^{2} \right) G(t, X_{t}, X_{t}', C_{t}, C_{t}') dB_{t} \\ & + \psi_{\eta}' \left(\left(X_{t} - X_{t}' \right)^{2} \right) G^{2}(t, X_{t}, X_{t}', C_{t}, C_{t}') dt \\ & + 2 (X_{t} - X_{t}')^{2} \psi_{\eta}'' \left(\left(X_{t} - X_{t}' \right)^{2} \right) G^{2}(t, X_{t}, X_{t}', C_{t}, C_{t}') dt, \end{split}$$

which is just another way of writing that for all $t \geq 0$

$$\begin{split} \psi_{\eta} \left(\left(X_{t} - X_{t}' \right)^{2} \right) &= \psi_{\eta} \left(\left(X_{0} - X_{0}' \right)^{2} \right) \\ &+ \int_{0}^{t} 2(X_{s} - X_{s}') \psi_{\eta}' \left(\left(X_{s} - X_{s}' \right)^{2} \right) F(s, X_{s}, X_{s}', C_{s}, C_{s}') ds \\ &+ \int_{0}^{t} 2(X_{s} - X_{s}') \psi_{\eta}' \left(\left(X_{s} - X_{s}' \right)^{2} \right) G(s, X_{s}, X_{s}', C_{s}, C_{s}') dB_{s} \\ &+ \int_{0}^{t} \left(\psi_{\eta}' \left(\left(X_{s} - X_{s}' \right)^{2} \right) + 2(X_{s} - X_{s}')^{2} \psi_{\eta}'' \left(\left(X_{s} - X_{s}' \right)^{2} \right) \right) G^{2}(s, X_{s}, X_{s}', C_{s}, C_{s}') ds \end{split}$$

We now compute the limit of each term. First

$$\psi_{\eta}\left(\left(X_{t}-X_{t}^{\prime}\right)^{2}\right)\xrightarrow[\eta\rightarrow0]{}|X_{t}-X_{t}^{\prime}|\quad\text{and}\quad\psi_{\eta}\left(\left(X_{0}-X_{0}^{\prime}\right)^{2}\right)\xrightarrow[\eta\rightarrow0]{}|X_{0}-X_{0}^{\prime}|.$$

Then

$$\left| 2(X_s - X_s')\psi_{\eta}' \left((X_s - X_s')^2 \right) \right| = \frac{|X_s - X_s'|}{\sqrt{(X_s - X_s')^2 + \eta}} \le 1.$$

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Thus for all $t \ge 0$, by dominated convergence (recall F is a continuous function, thus integrable on [0,t]), we obtain almost surely

$$\int_0^t 2(X_s - X_s')\psi_\eta' \left((X_s - X_s')^2 \right) F(s, X_s, X_s', C_s, C_s') ds$$

$$\xrightarrow{\eta \to 0} \int_0^t \operatorname{sign}(X_s - X_s') F(s, X_s, X_s', C_s, C_s') ds,$$

and by Theorem 2.12 Chapter 4 of [26], almost surely we have

$$\int_0^t 2(X_s - X_s')\psi_\eta' \left((X_s - X_s')^2 \right) G(s, X_s, X_s', C_s, C_s') dB_s$$

$$\xrightarrow{\eta \to 0} \int_0^t \operatorname{sign}(X_s - X_s') G(s, X_s, X_s', C_s, C_s') dB_s,$$

Finally, since $G(s,X_s,X_s^\prime,C_s,C_s^\prime)=0$ if $|X_s-X_s^\prime|< R_G$ and

$$\psi_{\eta}'\left((X_{s}-X_{s}')^{2}\right)+2(X_{s}-X_{s}')^{2}\psi_{\eta}''\left((X_{s}-X_{s}')^{2}\right)$$

$$=\frac{1}{2}\left(\frac{1}{\sqrt{(X_{s}-X_{s}')^{2}+\eta}}-\frac{(X_{s}-X_{s}')^{2}}{\left((X_{s}-X_{s}')^{2}+\eta\right)^{3/2}}\right)$$

$$=\frac{1}{2}\frac{\eta}{\left((X_{s}-X_{s}')^{2}+\eta\right)^{3/2}}$$

$$\leq \frac{1}{2}\frac{\eta}{|X_{s}-X_{s}'|^{3}},$$

by dominated convergence we almost surely have

$$\int_{0}^{t} \left(\psi_{\eta}' \left((X_s - X_s')^2 \right) + 2(X_s - X_s')^2 \psi_{\eta}'' \left((X_s - X_s')^2 \right) \right) G^2(s, X_s, X_s', C_s, C_s') ds \xrightarrow[\eta \to 0]{} 0$$

Thus for all $t \ge 0$ we almost surely have (A.1), and continuity allows us to conclude that we almost surely have for all $t \ge 0$ (A.1).

A.2 On Lemma 2.1

Lemma A.2. For all
$$z=(x,c), z'=(x',c')\in\mathbb{R}^d$$
, denoting $r(z,z')=|x-x'|+\delta|c-c'|$
$$r(z,z')^2\leq \frac{16(1+\delta^2)}{\min{(\gamma,1)}}\left(H(z)+H(z')\right)\,, \tag{A.2}$$

so that, in particular, for any constant B>0, if $r(z,z')\geq R=\sqrt{\frac{1280(1+\delta^2)B}{\lambda \min(\gamma,1)}}$, then

$$\lambda H(z) + \lambda H(z') > 80B.$$

Proof. We have $H(z) \ge \frac{\gamma}{4} x^2 + \frac{c^2}{4} \ge \frac{1}{4} \min{(\gamma, 1)} \left(x^2 + c^2\right)$. Thus

$$\begin{split} r(z,z')^2 = & (|x-x'| + \delta|c-c'|)^2 \\ \leq & 4(1+\delta^2)(x^2+c^2) + 4(1+\delta^2)(x'^2+c'^2) \\ \leq & 16\frac{(1+\delta^2)}{\min{(\gamma,1)}} (H(z) + H(z')) \end{split}$$

A.3 Proof of Lyapunov's property of H and its consequences Lyapunov's property

Proof of Lemma 2.2. We write the proof for (2.3), as it also yields (2.4) by considering μ to be the empirical measure. We notice

$$\partial_C H = c + \alpha$$
 and $\partial_X H = \gamma x + \beta$,

SO

$$\mathcal{L}_{\mu}H(z) = \partial_{x}H(z)(x - x^{3}) + \partial_{x}H(z)K_{X} * \mu(z) - c\partial_{c}H(z)$$

$$+ \partial_{c}H(z)K_{C} * \mu(z) + \frac{\sigma_{X}^{2}\gamma}{2} + \frac{\sigma_{C}^{2}}{2}$$

$$= (\gamma x + \beta)(x - x^{3}) - c(c + \alpha) + (\gamma x + \beta)K_{X} * \mu(z)$$

$$+ (c + \alpha)K_{C} * \mu(z) + \frac{\sigma_{X}^{2}\gamma}{2} + \frac{\sigma_{C}^{2}}{2}.$$

First, we focus on the interaction terms. We have

$$|K_X * \mu(z)| \le \int_{\mathbb{R}^2} |K_X(z - z')| \mu(dz')$$

$$\le \int_{\mathbb{R}^2} L_X(\|z\|_1 + \|z'\|_1) \mu(dz').$$

Hence,

$$(\gamma x + \beta)K_X * \mu(z)$$

$$\leq L_X(\gamma |x| + \beta)(|x| + |c| + \mathbb{E}_{\mu}(|X|) + \mathbb{E}_{\mu}(|C|))$$

$$\leq L_X(\gamma |x|^2 + \gamma |x||c| + \gamma |x|\mathbb{E}_{\mu}(|X|) + \gamma |x|\mathbb{E}_{\mu}(|C|) + \beta |x| + \beta |c| + \beta \mathbb{E}_{\mu}(|X|)$$

$$+\beta \mathbb{E}_{\mu}(|C|)),$$

and using Young's inequality $ab \leq \frac{\alpha}{2}a^2 + \frac{1}{2\alpha}b^2$ ($\alpha=16$ when we separate the x and c terms, and $\alpha=1$ otherwise on the various terms) we get

$$\begin{split} &(\gamma x + \beta)K_X * \mu(z) \\ & \leq & L_X \Big(\gamma |x|^2 + 8\gamma^2 |x|^2 + \frac{|c|^2}{32} + \frac{\gamma}{2} |x|^2 + \frac{\gamma}{2} \mathbb{E}_{\mu} (|X|)^2 + 8\gamma^2 |x|^2 + \frac{\mathbb{E}_{\mu} (|C|)^2}{32} + \frac{\beta^2}{2} \\ & \quad + \frac{|x|^2}{2} + 8\beta^2 + \frac{|c|^2}{32} + \frac{\beta^2}{2} + \frac{1}{2} \mathbb{E}_{\mu} (|X|)^2 + 8\beta^2 + \frac{\mathbb{E}_{\mu} (|C|)^2}{32} \Big) \\ & = & L_X \left(17\beta^2 + |x|^2 \left(\frac{1}{2} + \frac{3}{2} \gamma + 16\gamma^2 \right) + \frac{|c|^2}{16} + \mathbb{E}_{\mu} (|X|)^2 \left(\frac{\gamma}{2} + \frac{1}{2} \right) + \frac{\mathbb{E}_{\mu} (|C|)^2}{16} \right). \end{split}$$

Likewise

$$(c+\alpha)K_C * \mu(z) \le L_C \left(17\alpha^2 + \frac{17}{2}|x|^2 + |c|^2 \left(\frac{3}{2} + \frac{3}{32}\right) + \frac{17}{2}\mathbb{E}_{\mu}(|X|)^2 + \mathbb{E}_{\mu}(|C|)^2 \left(\frac{1}{2} + \frac{1}{32}\right)\right).$$

The idea is to bound $\lambda H(z) + \mathcal{L}_{\mu}H(z)$, by distinguishing 3 types of terms: we isolate terms in $\mathbb{E}_{\mu}(|C|)^2 - c^2$, $\mathbb{E}_{\mu}(|X|)^2 - x^2$, and we group polynomial terms. Then, we notice

the polynomial is upper bounded by a constant A. Thus

$$\lambda H(z) + \mathcal{L}_{\mu} H(z) - \frac{\sigma_{X}^{2} \gamma}{2} - \frac{\sigma_{C}^{2}}{2}$$

$$= \lambda \left(\frac{1}{2} \gamma x^{2} + \beta x + \frac{1}{2} c^{2} + \alpha c + H_{0} \right) + (\gamma x + \beta)(x - x^{3}) - c(c + \alpha)$$

$$+ (\gamma x + \beta) K_{X} * \mu(z) + (c + \alpha) K_{C} * \mu(z)$$

$$\leq (\lambda H_{0} + 17 \beta^{2} L_{X} + 17 \alpha^{2} L_{C}) - \gamma x^{4} - \beta x^{3} + (1 + \lambda) \beta x$$

$$+ \left((1 + \frac{\lambda}{2}) \gamma + L_{X} \left(1 + 2 \gamma + 16 \gamma^{2} \right) + 17 L_{C} \right) x^{2}$$

$$+ \left(\frac{L_{C}}{8} + L_{C} \left(2 + \frac{1}{8} \right) - \left(1 - \frac{\lambda}{2} \right) \right) c^{2} - (1 - \lambda) \alpha c$$

$$+ \left(\frac{L_{X}}{16} + \frac{L_{C}}{2} + \frac{L_{C}}{32} \right) \left(\mathbb{E}_{\mu} (|C|)^{2} - c^{2} \right)$$

$$+ \left(\frac{\gamma}{2} L_{X} + \frac{1}{2} L_{X} + \frac{17}{2} L_{C} \right) \left(\mathbb{E}_{\mu} (|X|)^{2} - x^{2} \right).$$

Provided that

$$\frac{L_X}{8} + L_C\left(2 + \frac{1}{8}\right) < 1 - \frac{\lambda}{2},$$

there is A > 0 such that

$$-\gamma x^{4} - \beta x^{3} + (1+\lambda)\beta x + \left((1+\frac{\lambda}{2})\gamma + L_{X} \left(1 + 2\gamma + 16\gamma^{2} \right) + 17L_{C} \right) x^{2} + \left(\frac{L_{C}}{8} + L_{C} \left(2 + \frac{1}{8} \right) - \left(1 - \frac{\lambda}{2} \right) \right) c^{2} - (1-\lambda)\alpha c \le A.$$

Hence the result

$$\mathcal{L}_{\mu}H(\bar{z}) \leq B + (\alpha_X L_X + \beta_X L_C) \left(\mathbb{E}_{\mu}(|X|)^2 - \bar{x}^2 \right)$$
$$+ (\alpha_C L_X + \beta_C L_C) \left(\mathbb{E}_{\mu}(|C|)^2 - \bar{c}^2 \right) - \lambda H(\bar{z}).$$

First consequences

Proof of Proposition 2.3. Inequality (2.5) simply relies on the sum of (2.4) for each i and the fact that $\mathcal{L}^{j,N}\left(H\left(Z_t^{i,N}\right)\right)=0$ for $i\neq j$

$$\begin{split} &\frac{1}{N} \sum_{i=1}^{N} \mathcal{L}^{N} \left(H \left(Z_{t}^{i,N} \right) \right) = \frac{1}{N} \sum_{i=1}^{N} \mathcal{L}^{i,N} \left(H \left(Z_{t}^{i,N} \right) \right) \\ &\leq \frac{1}{N} \sum_{i=1}^{N} \left[B + \left(\alpha_{X} L_{X} + \beta_{X} L_{C} \right) \left(\left(\frac{1}{N} \sum_{k=1}^{N} |X_{t}^{k,N}| \right)^{2} - \left(X_{t}^{i,N} \right)^{2} \right) \\ &+ \left(\alpha_{C} L_{X} + \beta_{C} L_{C} \right) \left(\left(\frac{1}{N} \sum_{k=1}^{N} |C_{t}^{k,N}| \right)^{2} - \left(C_{t}^{i,N} \right)^{2} \right) - \lambda H \left(Z_{t}^{i,N} \right) \right] \\ &\leq B - \lambda \frac{1}{N} \sum_{i=1}^{N} H \left(Z_{t}^{i,N} \right). \end{split}$$

The last inequality uses the fact that $\left(\frac{1}{N}\sum_{i=1}^{N}|y_i|\right)^2-\frac{1}{N}\sum_{i=1}^{N}\left(y_i\right)^2\leq 0$ for all $(y_i)_{1\leq i\leq N}\in\mathbb{R}^N$.

Bounds on the second moments of processes We can now prove the uniform in bounds on the second moments of $X_t^{i,N}$, $C_t^{i,N}$, \bar{X}_t^i and \bar{C}_t^i from (2.3) and (2.4). Let's notice that $(X_t^{i,N,\kappa},C_t^{i,N,\kappa})_i$ coincides with $(X_t^{i,N},C_t^{i,N})_i$ before the time T_κ defined in Subsection 1.3. Since our interest is in $(X_t^{i,N},C_t^{i,N})_i$, we chose to give the proof of the Proposition 1.6. The proof of the Lemma 1.5 is very similar.

Proof of Proposition 1.6. K_X and K_C are Lipschitz with constants L_X and L_C respectively. We do not assume any bounds on these constants. We assume for each $i \leq N$, $\mathbb{E}(|X_0^{i,N}|^2) < +\infty$ and $\mathbb{E}(|C_0^{i,N}|^2) < +\infty$. We have

$$d\left(\frac{e^{\lambda t}}{N}\sum_{i=1}^{N}H\left(Z_{t}^{i,N}\right)\right) = \lambda \frac{e^{\lambda t}}{N}\sum_{i=1}^{N}H\left(Z_{t}^{i,N}\right)dt + e^{\lambda t}\mathcal{L}^{N}\left(\frac{1}{N}\sum_{i=1}^{N}H\left(Z_{t}^{i,N}\right)\right)dt + dM_{t},$$

where M_t is a local martingale. Using (2.4)

$$d\left(\frac{e^{\lambda t}}{N}\sum_{i=1}^{N}H\left(Z_{t}^{i,N}\right)\right) = A_{t}dt + dM_{t},$$

where $A_t \leq Be^{\lambda t}$. Let τ_n be an increasing sequence of localizing stopping times converging to ∞ for M_t

$$\mathbb{E}\left(\frac{e^{\lambda t \wedge \tau_n}}{N} \sum_{i=1}^{N} H\left(Z_{t \wedge \tau_n}^{i,N}\right)\right) \leq \mathbb{E}\left(\frac{1}{N} \sum_{i=1}^{N} H\left(Z_0^{i,N}\right)\right) + \mathbb{E}\left(\int_0^{t \wedge \tau_n} B e^{\lambda s} ds\right)$$

$$\leq \mathbb{E}\left(\frac{1}{N} \sum_{i=1}^{N} H\left(Z_0^{i,N}\right)\right) + B \frac{\mathbb{E}\left(e^{\lambda t \wedge \tau_n}\right) - 1}{\lambda}$$

$$\leq \mathbb{E}\left(\frac{1}{N} \sum_{i=1}^{N} H\left(Z_0^{i,N}\right)\right) + B \max\left(\frac{e^{\lambda t} - 1}{\lambda}, \frac{1}{|\lambda|}\right),$$

where the maximum on this last inequality depends on the sign of λ . By Fatou's lemma, we obtain

$$\begin{split} e^{\lambda t} \mathbb{E} \left(\frac{1}{N} \sum_{i=1}^{N} H \left(Z_{t}^{i,N} \right) \right) = & \mathbb{E} \left(\liminf_{n \to \infty} \frac{e^{\lambda t \wedge \tau_{n}}}{N} \sum_{i=1}^{N} H \left(Z_{t \wedge \tau_{n}}^{i,N} \right) \right) \\ \leq & \liminf_{n \to \infty} \mathbb{E} \left(\frac{e^{\lambda t \wedge \tau_{n}}}{N} \sum_{i=1}^{N} H \left(Z_{t \wedge \tau_{n}}^{i,N} \right) \right) \\ \leq & \mathbb{E} \left(\frac{1}{N} \sum_{i=1}^{N} H \left(Z_{0}^{i,N} \right) \right) + B \max \left(\frac{e^{\lambda t} - 1}{\lambda}, \frac{1}{|\lambda|} \right). \end{split}$$

Hence the various bounds on $\mathbb{E}\left(|X_t^{i,N}|^2\right)$ and $\mathbb{E}\left(|C_t^{i,N}|^2\right)$, since by Lemma 2.1 (i) we have

$$\begin{split} \mathbb{E} H\left(Z_t^{i,N}\right) &\geq \frac{\gamma}{4} \mathbb{E}\left(|X_t^{i,N}|^2\right) + \frac{1}{4} \mathbb{E}\left(|C_t^{i,N}|^2\right) \\ \text{and } \mathbb{E} H\left(Z_0^{i,N}\right) &\leq \gamma \mathbb{E}\left(|X_0^{i,N}|^2\right) + \mathbb{E}\left(|C_0^{i,N}|^2\right) + \frac{3}{2} H_0. \end{split}$$

These bounds are uniform in time provided $\lambda>0$, i.e $\frac{L_X}{8}+L_C\left(2+\frac{3}{32}\right)<1$.

Proof of Proposition 1.7 and Lemma 2.4. The proof is done in exactly the same way as the proof of Proposition 1.6 above using (2.3).

A.4 Proof of Lemma 2.6

We now prove that there are constants c, ϵ and δ such that

$$c + 2\epsilon \tilde{B} \le \frac{\sigma_X^2}{2} \left(\int_0^R \Phi(s) \phi(s)^{-1} ds \right)^{-1}$$
(A.3)

$$2c + 4\epsilon \tilde{B} \le \left(1 - L_C - \frac{1 + L_X}{\delta}\right) \min_{r \in (0,R]} \frac{f'(r)r}{f(r)} \tag{A.4}$$

$$c \le \frac{\lambda}{160} \frac{\frac{80\epsilon \tilde{B}}{\lambda}}{1 + \frac{80\epsilon \tilde{B}}{\lambda}} \tag{A.5}$$

$$\delta > \frac{1 + L_X}{1 - L_C} \tag{A.6}$$

• Since for all $u \geq 0$, $0 < \phi\left(u\right) \leq 1$, we have $0 < \Phi\left(s\right) = \int_0^s \phi\left(u\right) du \leq s$, i.e $s/\Phi\left(s\right) \geq 1$. Therefore

$$\inf_{r \in (0,R]} \frac{r\phi\left(r\right)}{\Phi\left(r\right)} \ge \inf_{r \in (0,R]} \phi\left(r\right) = \phi\left(R\right).$$

It is thus sufficient for (A.4) to have

$$2c + 4\epsilon \tilde{B} \le \frac{1}{2} \left(1 - L_C - \frac{1 + L_X}{\delta} \right) \phi(R).$$

• We have

$$\phi(r) \le \exp\left(-\frac{1}{4\sigma_X^2}r^2\right).$$

So

$$\Phi\left(r\right) \leq \int_{0}^{\infty} \exp\left(-\frac{r^{2}}{4\sigma_{X}^{2}}\right) dr = \sigma_{X}\sqrt{\pi}.$$

Then

$$\int_{0}^{R} \frac{\Phi(r)}{\phi(r)} dr \le \sigma_{X} \sqrt{\pi} R \frac{1}{\phi(R)}.$$

It is thus sufficient for (A.3) that

$$c + 2\epsilon \tilde{B} \le \frac{\sigma_X}{2\sqrt{\pi}} \frac{\phi(R)}{R}.$$

• The various conditions involving c invite us to consider $2\epsilon \tilde{B} = \eta c$. Then

$$\begin{split} c & \leq \frac{\lambda}{160} \frac{\frac{80\epsilon \tilde{B}}{\lambda}}{1 + \frac{80\epsilon \tilde{B}}{\lambda}} \iff c \leq \frac{\lambda}{160} \frac{40\eta c}{\lambda + 40\eta c} \\ & \iff 1 \leq \lambda \frac{\eta}{4\lambda + 160\eta c} \text{ (since } c \geq 0\text{)} \\ & \iff c \leq \frac{\lambda}{160} \frac{\eta - 4}{\eta}. \end{split}$$

· We choose to write

$$\delta = (1+\tilde{\delta})\frac{1+L_{X,\max}}{1-L_{C,\max}} > \frac{1+L_X}{1-L_C}$$

• Let us assume, for simplicity, that $\epsilon \leq 1$. It is sufficient for this later condition to have

$$c \leq \frac{2\tilde{B}}{\eta}$$
.

• The appearance of $\phi\left(R\right)$ suggests we should try to minimize it. We recall

$$\phi(r) = \exp\left(-\frac{1}{4\sigma_X^2} \left(1 + \delta\gamma + L_X + \delta L_C + (\epsilon C_{f,1} + C_{f,2}) \sigma_X^2\right) r^2\right)$$
$$\geq \exp\left(-\frac{1}{4\sigma_X^2} \left(1 + \delta\gamma + L_X + \delta L_C + (C_{f,1} + C_{f,2}) \sigma_X^2\right) r^2\right).$$

It is therefore sufficient for (A.3) to have

$$c \leq \frac{1}{1+\eta} \frac{\sigma_X}{2\sqrt{\pi}} \frac{1}{R} \exp\left(-\frac{1}{4\sigma_X^2} \left(1 + \delta\gamma + L_X + \delta L_C + \left(\mathcal{C}_{f,1} + \mathcal{C}_{f,2}\right) \sigma_X^2\right) R^2\right),$$

and for (A.4) to have

$$c \leq \frac{1}{2(1+\eta)} \left(1 - L_C - \frac{1+L_X}{\delta} \right) \times \exp\left(-\frac{R^2}{4\sigma_X^2} \left(1 + \delta\gamma + L_X + \delta L_C + \left(\mathcal{C}_{f,1} + \mathcal{C}_{f,2} \right) \sigma_X^2 \right) \right).$$

• Finally, we bound L_X and L_C by either 0 or $L_{X,\max}$ and $L_{C,\max}$, to obtain bounds on c independent of L_X and L_C .

A.5 Proof of Lemma 2.7

Let $z, z' \in \mathbb{R}^2$.

Proof of control of the L^1 **distance:** We have

$$||z - z'||_1 = |x - x'| + |c - c'| \le \frac{1}{\min(\delta, 1)} (|x - x'| + \delta |c - c'|) = \frac{1}{\min(\delta, 1)} r(z, z').$$

If $r(z,z') \leq 1 \leq R$, we have, using Lemma 2.6

$$r(z, z') \le \frac{f(r)}{f'(R)} \le \frac{f(r)}{\phi(R)g(R)} \left(1 + \epsilon \tilde{H}(z) + \epsilon \tilde{H}(z')\right).$$

If $r(z, z') \ge 1$, we have, using (A.2)

$$\begin{split} r(z,z') &\leq r(z,z')^2 \\ &\leq \frac{16(1+\delta^2)}{\epsilon \min{(\gamma,1)}} \left(\epsilon H(z) + \epsilon H(z')\right) \\ &\leq \frac{16(1+\delta^2)}{\epsilon \min{(\gamma,1)}} \frac{f(r)}{f(1)} \left(1 + \epsilon H(z) + \epsilon H(z')\right) \\ &\leq \frac{16(1+\delta^2)}{\epsilon \min{(\gamma,1)}} \frac{f(r)}{\phi(R)g(R)} \left(1 + \epsilon \tilde{H}(z) + \epsilon \tilde{H}(z')\right). \end{split}$$

Thus

$$||z-z'||_1 \leq \frac{1}{\min(\delta,1)} \frac{1}{\phi(R)q(R)} \max\left(\frac{16(1+\delta^2)}{\epsilon \min(\gamma,1)}, 1\right) f(r(z,z')) \left(1+\epsilon \tilde{H}(z)+\epsilon \tilde{H}(z')\right).$$

Proof of control of the L^2 **distance:** We have

$$r(z, z')^{2} = (|x - x'| + \delta|c - c'|)^{2} \ge |x - x'|^{2} + \delta^{2}|c - c'|^{2}$$

$$\ge \min(1, \delta^{2}) (|x - x'|^{2} + |c - c'|^{2}).$$

If $r(z, z') \ge 1$, we have, using (A.2)

$$r(z, z')^2 \le \frac{16(1+\delta^2)}{\epsilon \min(\gamma, 1)} \frac{f(r)}{\phi(R)g(R)} \left(1 + \epsilon \tilde{H}(z) + \epsilon \tilde{H}(z')\right).$$

If $r(z, z') \le 1 \le R$, we have, using Lemma 2.6

$$r(z,z')^2 \le r(z,z') \le \frac{f(r)}{f'_-(R)} \le \frac{f(r)}{\phi(R)g(R)} \left(1 + \epsilon \tilde{H}(z) + \epsilon \tilde{H}(z')\right)$$

Thus

$$||z - z'||_2^2 \le \frac{1}{\min(\delta^2, 1)} \frac{1}{\phi(R)g(R)} \max\left(\frac{16(1 + \delta^2)}{\epsilon \min(\gamma, 1)}, 1\right) f(r(z, z')) \left(1 + \epsilon \tilde{H}(z) + \epsilon \tilde{H}(z')\right).$$

Proof of the second control of the L^1 **distance:** We have, if $r(z, z') \le 1 \le R$

$$r(z, z') \le \frac{f(r)}{f'_{-}(R)} \le \frac{f(r)}{\phi(R)g(R)} \left(1 + \epsilon \sqrt{H(z)} + \epsilon \sqrt{H(z')}\right).$$

and, if $r(z, z') \ge 1$, recall Lemma 2.1.

$$||z - z'||_1 \le \sqrt{\frac{4}{\gamma}H(z)} + \sqrt{\frac{4}{\gamma}H(z')} + \sqrt{4H(z)} + \sqrt{4H(z')}$$

$$\le 4 \max\left(\sqrt{\frac{1}{\gamma}}, 1\right) \left(\sqrt{H(z)} + \sqrt{H(z')}\right)$$

$$\le \frac{4}{\epsilon} \max\left(\sqrt{\frac{1}{\gamma}}, 1\right) \frac{f(r)}{\phi(R)a(R)} \left(1 + \epsilon\sqrt{H(z)} + \epsilon\sqrt{H(z')}\right),$$

and thus

$$||z - z'||_1 \le \frac{1}{\phi(R)g(R)} \max\left(1, \frac{4}{\epsilon} \max\left(\sqrt{\frac{1}{\gamma}}, 1\right)\right) f(r(z, z')) \left(1 + \epsilon \sqrt{H(z)} + \epsilon \sqrt{H(z')}\right).$$

Independence with respect to L_X and L_C The a priori bounds $L_X \in [0, L_{X, \max}]$ and $L_C \in [0, L_{C, \max}]$ allow us to bound $\phi(R)$ independently of L_C and L_X by ϕ_{\min} (and we also use $g(R) \geq \frac{1}{2}$), thus giving us constant \mathcal{C}_1 , \mathcal{C}_2 and \mathcal{C}_z independent of L_C and L_X .

A.6 Proof of Lemmas 3.2 and 3.7

Proof of Lemma 3.2. Let's prove there exists a uniform in time bound on $\mathbb{E}(G_t^i)$ and $\mathbb{E}[(G_t^i)^2]$. First, let's recall the definition of G from (2.26)

$$G_t^i = 1 + \epsilon \tilde{H}(\bar{Z}_t^i) + \epsilon \tilde{H}(Z_t^{i,N}) + \frac{\epsilon}{N} \sum_{i=1}^N \tilde{H}(Z_t^{j,N}) + \frac{\epsilon}{N} \sum_{i=1}^N \tilde{H}(\bar{Z}_t^j).$$

The idea is to bound the different expectations in terms of the expectations at time t=0. Since $\mathbb{E}(e^{\tilde{a}(|X_0|+|C_0|)})$ is finite, we know that for each $k\in\mathbb{N}$, $\mathbb{E}(|X_0|^k)$ and $\mathbb{E}(|C_0|^k)$ are

also finite. We deduce that for each $k \in \mathbb{N}$, for each $j \leq N$, $\mathbb{E}[H(\bar{Z}_0^j)^k]$ and $\mathbb{E}[H(Z_0^{j,N})^k]$ are finite.

In fact, to bound uniformly in time the first moment, we only have to bound $\mathbb{E}(\tilde{H}(Z_t^{j,N}))$ and $\mathbb{E}(\tilde{H}(\bar{Z}_t^j))$ for each $j \leq N$. Let's begin with \bar{Z}^j . By (2.16), we have

$$\frac{d}{dt}\mathbb{E}\left[\tilde{H}\left(\bar{Z}_{t}^{j}\right)\right] \leq \tilde{B} - \frac{\lambda}{4}\mathbb{E}\left[\tilde{H}\left(\bar{Z}_{t}^{j}\right)\right].$$

By using Itô's formula on $e^{\lambda t/4} ilde{H} \left(ar{Z}_t^j
ight)$ and the bound above, we obtain

$$\mathbb{E}\left[\tilde{H}\left(\bar{Z}_{t}^{j}\right)\right] \leq \frac{4\tilde{B}}{\lambda} + e^{-\frac{\lambda}{4}t} \left(\mathbb{E}\left[\tilde{H}\left(\bar{Z}_{0}^{j}\right)\right] - \frac{4\tilde{B}}{\lambda}\right)$$
$$\leq \max\left(\mathbb{E}\left[\tilde{H}\left(\bar{Z}_{0}^{j}\right)\right], \frac{4\tilde{B}}{\lambda}\right).$$

By (2.9), in Lemma 2.5, we deduce the following inequality and we apply Cauchy-Schwarz inequality

$$\mathbb{E}\left[\tilde{H}\left(\bar{Z}_{0}^{j}\right)\right] \leq \mathbb{E}\left[H\left(\bar{Z}_{0}^{j}\right) \exp\left(a\sqrt{H\left(\bar{Z}_{0}^{j}\right)}\right)\right]$$

$$\leq \mathbb{E}\left[H\left(\bar{Z}_{0}^{j}\right)^{2}\right]^{1/2} \mathbb{E}\left[\exp\left(2a\sqrt{H\left(\bar{Z}_{0}^{j}\right)}\right)\right]^{1/2}.$$
(A.7)

We already know $\mathbb{E}\left[H\left(\bar{Z}_0^j\right)^2\right]$ is bounded. Now, it is enough to prove that there exist C such that for all $z\in\mathbb{R}^2$

$$\exp\left(2a\sqrt{H\left(z\right)}\right) \le C \times e^{\tilde{a}(|x|+|c|)}.$$

In fact, from the definition of H in (2.1), we have

$$2\sqrt{H(z)} = \sqrt{2}\sqrt{\gamma\left(x + \frac{\beta}{\gamma}\right)^2 + (c + \alpha)^2 + H_0}$$

$$\leq \sqrt{2\gamma}\left|x + \frac{\beta}{\gamma}\right| + \sqrt{2}\left|c + \alpha\right| + \sqrt{H_0}$$

$$\leq \sqrt{2\gamma}|x| + \sqrt{2}|c| + \frac{1}{a}\ln C,$$

where C is a constant independent of z. Finally, since $\max(a\sqrt{2\gamma},a\sqrt{2}) \leq \tilde{a}$, we have

$$\exp\left(2a\sqrt{H\left(z\right)}\right) \le C \times e^{\tilde{a}(|x|+|c|)}.$$

Then, $\mathbb{E}\left[\exp\left(2a\sqrt{H\left(\bar{Z}_0^j\right)}\right)\right]$ is bounded and we deduce $\mathbb{E}(\tilde{H}(\bar{Z}_t^j))$ is bounded for each j < N and all t > 0.

The same calculations can be done for $Z_t^{j,N}$. By (2.19), we have

$$\mathcal{L}^N\left(\frac{1}{N}\sum_{i=1}^N \tilde{H}(Z_t^{i,N})\right) \leq \tilde{B} - \frac{\lambda}{4}\left(\frac{1}{N}\sum_{i=1}^N \tilde{H}(Z_t^{i,N})\right).$$

In particular,

$$\frac{d}{dt}\left[\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{H}(Z_{t}^{i,N})\right)\right] \leq \mathbb{E}\left[\mathcal{L}^{N}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{H}(Z_{t}^{i,N})\right)\right] \leq \tilde{B} - \frac{\lambda}{4}\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{H}(Z_{t}^{i,N})\right),$$

and we can use the same method as above.

Finally, we have proved that for each $j \leq N$, $\mathbb{E}(\tilde{H}(Z_t^{j,N}))$ and $\mathbb{E}(\tilde{H}(\bar{Z}_t^j))$ are bounded uniformly in time. Thus, $\mathbb{E}(G_t^i)$ is bounded uniformly in time (and in N).

To bound the second moment of G_t^i , we have to bound each type of the following expectations $\mathbb{E}[\tilde{H}(Z_t^{j_1,N})\tilde{H}(Z_t^{j_2,N})]$, $\mathbb{E}[\tilde{H}(Z_t^{j_1,N})\tilde{H}(\bar{Z}_t^{j_2})]$, $\mathbb{E}[\tilde{H}(\bar{Z}_t^{j_1})\tilde{H}(\bar{Z}_t^{j_2})]$, $\mathbb{E}[\tilde{H}(Z_t^{j,N})^2]$ and $\mathbb{E}[\tilde{H}(\bar{Z}_t^{j_1})^2]$. By Cauchy-Schwarz inequality, it is in fact enough to bound $\mathbb{E}[\tilde{H}(Z_t^{j,N})^2]$ and $\mathbb{E}[\tilde{H}(\bar{Z}_t^{j_1})^2]$.

First, by the definition of \tilde{H} in (2.8),

$$\tilde{H}(z)^{2} = \left(\frac{2}{a^{2}} \exp\left(a\sqrt{H(z)}\right) \left(a\sqrt{H(z)} - 1\right) + \frac{2}{a^{2}}\right)^{2}$$

$$\leq 2\frac{2^{2}}{a^{4}} \exp\left(2a\sqrt{H(z)}\right) \left(a\sqrt{H(z)} - 1\right)^{2} + 2\frac{2^{2}}{a^{4}}$$

$$\leq \frac{8}{a^{4}} \exp\left(2a\sqrt{H(z)}\right) \left(2a^{2}H(z) + 2\right) + \frac{8}{a^{4}}.$$

As for the first moment, the study of $Z_t^{j,N}$ is very similar to the one of \bar{Z}_t^j . Here, we only focus on the second one.

Using Cauchy-Schwarz inequality, bounds on
$$\mathbb{E}\left[H(\bar{Z}_t^j)^2\right]$$
 and $\mathbb{E}\left[\exp\left(4a\sqrt{H(\bar{Z}_t^j)}\right)\right]$

are sufficient to bound $\mathbb{E}[\tilde{H}(\bar{Z}_t^j)^2]$. The latter has already been bounded uniformly in time, and the former can be obtained by the same calculations as previously, replacing a by 4a (and thus assuming $\tilde{a} \geq 4\sqrt{2}a \max(\sqrt{\gamma}, 1)$, which we do).

Finally, we deduce
$$\mathbb{E}\left(\left(G_t^i\right)^2\right)$$
 is bounded uniformly in time.

Proof of Lemma 3.7. Using $\partial_X H(z) = \gamma x + \beta$, we have

$$\begin{split} \left| \partial_X \tilde{H}(Z_t^{i,N}) - \partial_X \tilde{H}(\bar{Z}_t^i) \right| \\ &= \left| \left(\gamma X_t^{i,N} + \beta \right) \exp \left(a \sqrt{H(Z_t^{i,N})} \right) - \left(\gamma \bar{X}_t^i + \beta \right) \exp \left(a \sqrt{H(\bar{Z}_t^i)} \right) \right| \\ &\leq \left| \gamma X_t^{i,N} - \gamma \bar{X}_t^i \right| \left(\exp \left(a \sqrt{H(Z_t^{i,N})} \right) + \exp \left(a \sqrt{H(\bar{Z}_t^i)} \right) \right) \\ &+ \left| \gamma \bar{X}_t^i + \beta \right| \left| \exp \left(a \sqrt{H(Z_t^{i,N})} \right) - \exp \left(a \sqrt{H(\bar{Z}_t^i)} \right) \right|. \end{split}$$

Since $\left|X_t^{i,N} - \bar{X}_t^i\right| \le r_t^i$,

$$\begin{split} \left| \gamma X_t^{i,N} - \gamma \bar{X}_t^i \right| \left(\exp\left(a \sqrt{H(Z_t^{i,N})} \right) + \exp\left(a \sqrt{H(\bar{Z}_t^i)} \right) \right) \\ \leq & \gamma r_t^i \left(\exp\left(a \sqrt{H(Z_t^{i,N})} \right) + \exp\left(a \sqrt{H(\bar{Z}_t^i)} \right) \right) \end{split}$$

By Lemma 2.1 (ii), we have $H(z) \geq \frac{1}{2} \min\left(\frac{1}{\gamma}, 1\right) (\gamma x + \beta)^2$. By the mean value theorem, for all $y_1 \leq y_2$ in \mathbb{R} , there exists $y_3 \in [y_1, y_2]$ such that $e^{ay_1} - e^{ay_2} = a(y_1 - y_2)e^{ay_3}$. In

particular, we have the following control $|e^{ay_1} - e^{ay_2}| \le a|y_1 - y_2|(e^{ay_1} + e^{ay_2})$. Thus

$$\begin{split} \left| \gamma \bar{X}_t^i + \beta \right| \left| \exp \left(a \sqrt{H(Z_t^{i,N})} \right) - \exp \left(a \sqrt{H(\bar{Z}_t^i)} \right) \right| \\ & \leq a \sqrt{\frac{2H(\bar{Z}_t^i)}{\min \left(\frac{1}{\gamma}, 1 \right)}} \left| \sqrt{H(Z_t^{i,N})} - \sqrt{H(\bar{Z}_t^i)} \right| \left(\exp \left(a \sqrt{H(Z_t^{i,N})} \right) + \exp \left(a \sqrt{H(\bar{Z}_t^i)} \right) \right) \\ & \leq a \sqrt{2 \max \left(\gamma, 1 \right)} \left| H(Z_t^{i,N}) - H(\bar{Z}_t^i) \right| \left(\exp \left(a \sqrt{H(Z_t^{i,N})} \right) + \exp \left(a \sqrt{H(\bar{Z}_t^i)} \right) \right). \end{split}$$

Then by the definition of H we get

$$\begin{split} \left| H(Z_t^{i,N}) \ - H(\bar{Z}_t^i) \right| \\ &= \left| \frac{1}{2} \gamma \left(\left(X_t^{i,N} \right)^2 - \left(\bar{X}_t^i \right)^2 \right) + \beta (X_t^{i,N} - \bar{X}_t^i) + \frac{1}{2} \left(\left(C_t^{i,N} \right)^2 - \left(\bar{C}_t^i \right)^2 \right) + \alpha (C_t^{i,N} - \bar{C}_t^i) \right| \\ &\leq \frac{1}{2} \gamma \left| X_t^{i,N} - \bar{X}_t^i \right| \left| X_t^{i,N} + \bar{X}_t^i \right| + \beta \left| X_t^{i,N} - \bar{X}_t^i \right| + \frac{1}{2} \left| C_t^{i,N} - \bar{C}_t^i \right| \left| C_t^{i,N} + \bar{C}_t^i \right| \\ &+ \alpha \left| C_t^{i,N} - \bar{C}_t^i \right| \,. \end{split}$$

Now, by Lemma 2.1 (i), we have $H(z) \geq \frac{\gamma}{4}x^2 + \frac{1}{4}c^2$ and since $\left|X_t^{i,N} - \bar{X}_t^i\right| \leq r_t^i$ and $\left|C_t^{i,N} - \bar{C}_t^i\right| \leq r_t^i/\delta$, we get

$$\left|X_t^{i,N} - \bar{X}_t^i\right| \left(\frac{1}{2}\gamma \left|X_t^{i,N} + \bar{X}_t^i\right| + \beta\right) \leq r_t^i \left(\sqrt{\gamma} \left(\sqrt{H(Z_t^{i,N})} + \sqrt{H(\bar{Z}_t^i)}\right) + \beta\right)$$

and

$$\left|C_t^{i,N} - \bar{C}_t^i\right| \left(\frac{1}{2} \left|C_t^{i,N} + \bar{C}_t^i\right| + \alpha\right) \leq \frac{r_t^i}{\delta} \left(\sqrt{H(Z_t^{i,N})} + \sqrt{H(\bar{Z}_t^i)} + \alpha\right).$$

Thus

$$\left|H(Z_t^{i,N}) - H(\bar{Z}_t^i)\right| \leq \left(\beta + \frac{\alpha}{\delta}\right)r_t^i + \left(\sqrt{\gamma} + \frac{1}{\delta}\right)r_t^i \left(\sqrt{H(Z_t^{i,N})} + \sqrt{H(\bar{Z}_t^i)}\right) + \frac{1}{\delta}r_t^i + \left(\sqrt{\gamma} + \frac{1}{\delta}\right)r_t^i + \frac{1}{\delta}r_t^i + \frac{1}{\delta}r_$$

Finally,

$$\begin{split} |\partial_X \tilde{H}(Z_t^{i,N}) - \partial_X \tilde{H}(\bar{Z}_t^i)| \\ \leq & \gamma r_t^i \left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right) + \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right) \right) \\ & + a\sqrt{2\max\left(\gamma,1\right)} \left(\beta + \frac{\alpha}{\delta}\right) r_t^i \left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right) + \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right) \right) \\ & + a\sqrt{2\max\left(\gamma,1\right)} \left(\sqrt{\gamma} + \frac{1}{\delta}\right) r_t^i \left(\sqrt{H(Z_t^{i,N})} + \sqrt{H(\bar{Z}_t^i)}\right) \\ & \times \left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right) + \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right) \right) \\ \leq & r_t^i \left(\gamma + a\sqrt{2\max\left(\gamma,1\right)} \left(\beta + \frac{\alpha}{\delta}\right)\right) \left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right) + \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right) \right) \\ & + ar_t^i \sqrt{2\max\left(\gamma,1\right)} \left(\sqrt{\gamma} + \frac{1}{\delta}\right) \\ & \times \left(2\sqrt{H(Z_t^{i,N})} \exp\left(a\sqrt{H(Z_t^{i,N})}\right) + 2\sqrt{H(\bar{Z}_t^i)} \exp\left(a\sqrt{H(\bar{Z}_t^i)}\right) \right). \end{split}$$

Now, we can finally use Lemma 2.5, and more precisely (2.9) and (2.10), we obtain

$$\begin{split} &|\partial_X \tilde{H}(Z_t^{i,N}) - \partial_X \tilde{H}(\bar{Z}_t^i)| \\ &\leq r_t^i \left(\gamma + a\sqrt{2\max\left(\gamma,1\right)} \left(\beta + \frac{\alpha}{\delta}\right)\right) \left(\tilde{H}(Z_t^{i,N}) + \tilde{H}(\bar{Z}_t^i) + \frac{4}{a^2} \left(\mathrm{e}^{a^2/2} - 1\right)\right) \\ &+ ar_t^i \sqrt{2\max\left(\gamma,1\right)} \left(\sqrt{\gamma} + \frac{1}{\delta}\right) \left(2a\tilde{H}(Z_t^{i,N}) + \frac{2}{a}(\mathrm{e} - 2) + 2a\tilde{H}(\bar{Z}_t^i) + \frac{2}{a}(\mathrm{e} - 2)\right) \\ &\leq r_t^i \left(\tilde{H}(Z_t^{i,N}) + \tilde{H}(\bar{Z}_t^i)\right) \left[\gamma + a\sqrt{2\max\left(\gamma,1\right)} \left(\beta + \frac{\alpha}{\delta}\right) + 2a^2\sqrt{2\max\left(\gamma,1\right)} \left(\sqrt{\gamma} + \frac{1}{\delta}\right)\right] \\ &+ r_t^i \left[\left(\gamma + a\sqrt{2\max\left(\gamma,1\right)} \left(\beta + \frac{\alpha}{\delta}\right)\right) \frac{4}{a^2} \left(\mathrm{e}^{a^2/2} - 1\right) \\ &+ 4\sqrt{2\max\left(\gamma,1\right)} \left(\sqrt{\gamma} + \frac{1}{\delta}\right) \left(\mathrm{e} - 2\right)\right]. \end{split}$$

We denote by $\mathcal{C}_{f,1}$ and $\mathcal{C}_{f,2}$ (given in Lemma 2.6) the following constants

$$C_{f,1} = 4 \left[\left(\gamma + a \sqrt{2 \max(\gamma, 1)} \left(\beta + \frac{\alpha}{\delta} \right) \right) \frac{4}{a^2} \left(e^{a^2/2} - 1 \right) + 4 \sqrt{2 \max(\gamma, 1)} \left(\sqrt{\gamma} + \frac{1}{\delta} \right) (e - 2) \right]$$

$$C_{f,2} = 4 \left[\gamma + a \sqrt{2 \max(\gamma, 1)} \left(\beta + \frac{\alpha}{\delta} \right) + 2a^2 \sqrt{2 \max(\gamma, 1)} \left(\sqrt{\gamma} + \frac{1}{\delta} \right) \right]$$

By the definition of G_t^i and since $G_t^i \geq 1$, we obtain

$$|\partial_X \tilde{H}(Z_t^{i,N}) - \partial_X \tilde{H}(\bar{Z}_t^i)| \le r_t^i \frac{G_t^i}{\epsilon} \frac{C_{f,2}}{4} + r_t^i G_t^i \frac{C_{f,1}}{4},$$

and eventually

$$\begin{split} 2\epsilon \left(1 + \frac{1}{N}\right) \sigma_X^2 \varphi_{\text{rc}} \Big(|X_t^{i,N} - \bar{X}_t^i|\Big)^2 \left|\partial_X \tilde{H}(Z_t^{i,N}) - \partial_X \tilde{H}(\bar{Z}_t^i)\right| \\ & \leq \left(\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2}\right) \sigma_X^2 \varphi_{\text{rc}} \Big(|X_t^{i,N} - \bar{X}_t^i|\Big)^2 r_t^i G_t^i. \end{split}$$

B Proof of Theorem 1.4 in the case $\sigma_X = 0$ and $\sigma_C > 0$

We quickly explain in this section how we may also deal with the case $\sigma_X=0$ and $\sigma_C>0$. Recall how the choice of the coupling method was motivated by the observation in (1.9) that the difference of potentials $\left|C_t^{i,N}-\bar{C}_t^i\right|$ was naturally contracting when $\left|X_t^{i,N}-\bar{X}_t^i\right|$ was close to 0. This lead us to use a reflection coupling on the Brownian motions acting on the potential X, to bring the difference close to 0, and it was thus necessary for σ_X to be positive (σ_C however did not matter). In the case $\sigma_X=0$, we then have to assume $\sigma_C>0$, and we do a change of variable, motivated by the following

observation. We have, when $\sigma_X = 0$

$$\begin{split} d(X_t^{i,N} - \bar{X}_t^i) &= \left((X_t^{i,N} - \bar{X}_t^i) - \left((X_t^{i,N})^3 - (\bar{X}_t^i)^3 \right) - \left(C_t^{i,N} - \bar{C}_t^i \right) \right) dt \\ &+ \left(\frac{1}{N} \sum_{j=1}^N K_X (Z_t^{i,N} - Z_t^{j,N}) - K_X * \bar{\rho}(\bar{Z}_t^i) \right) dt \\ &= \left(2(X_t^{i,N} - \bar{X}_t^i) - (C_t^{i,N} - \bar{C}_t^i) - (X_t^{i,N} - \bar{X}_t^i) - \left((X_t^{i,N})^3 - (\bar{X}_t^i)^3 \right) \right) dt \\ &+ \left(\frac{1}{N} \sum_{j=1}^N K_X (Z_t^{i,N} - Z_t^{j,N}) - K_X * \bar{\rho}(\bar{Z}_t^i) \right) dt. \end{split}$$

Thus

$$\begin{split} d|X_t^{i,N} - \bar{X}_t^i| = & \mathrm{sign}(X_t^{i,N} - \bar{X}_t^i) \left(2(X_t^{i,N} - \bar{X}_t^i) - (C_t^{i,N} - \bar{C}_t^i) \right) dt \\ & - \left(\left| (X_t^{i,N})^3 - (\bar{X}_t^i)^3 \right| - \left| X_t^{i,N} - \bar{X}_t^i \right| \right) dt \\ & + \mathrm{sign}(X_t^{i,N} - \bar{X}_t^i) \left(\frac{1}{N} \sum_{j=1}^N K_X(Z_t^{i,N} - Z_t^{j,N}) - K_X * \bar{\rho}(\bar{Z}_t^i) \right) dt. \end{split}$$

The quantity $|X_t^{i,N} - \bar{X}_t^i|$ is therefore naturally contracting when $|2(X_t^{i,N} - \bar{X}_t^i) - (C_t^{i,N} - \bar{C}_t^i)|$ is close to 0. Thanks to the presence of a Brownian motion in the stochastic differential equations defining the potential C, we can now use a reflection coupling to have $|2(X_t^{i,N} - \bar{X}_t^i) - (C_t^{i,N} - \bar{C}_t^i)|$ go to 0. Consider the following coupling

$$\begin{cases} dX_{t}^{i,N} = (X_{t}^{i,N} - (X_{t}^{i,N})^{3} - C_{t}^{i,N} - \alpha)dt + \frac{1}{N} \sum_{j=1}^{N} K_{X}(Z_{t}^{i,N} - Z_{t}^{j,N})dt \\ dC_{t}^{i,N} = (\gamma X_{t}^{i,N} - C_{t}^{i,N} + \beta)dt + \frac{1}{N} \sum_{j=1}^{N} K_{C}(Z_{t}^{i,N} - Z_{t}^{j,N})dt \\ + \sigma_{C}\varphi_{\text{sc}} \left(|2(X_{t}^{i,N} - \bar{X}_{t}^{i}) - (C_{t}^{i,N} - \bar{C}_{t}^{i})| \right) dB_{t}^{i,sc,C} \\ + \sigma_{C}\varphi_{\text{rc}} \left(|2(X_{t}^{i,N} - \bar{X}_{t}^{i}) - (C_{t}^{i,N} - \bar{C}_{t}^{i})| \right) dB_{t}^{i,rc,C}, \end{cases}$$
(B.1)

and

$$\begin{cases} d\bar{X}_{t}^{i} = (\bar{X}_{t}^{i} - (\bar{X}_{t}^{i})^{3} - \bar{C}_{t}^{i} - \alpha)dt + K_{X} * \bar{\rho}(\bar{Z}_{t}^{i})dt \\ d\bar{C}_{t}^{i} = (\gamma \bar{X}_{t}^{i} - \bar{C}_{t}^{i} + \beta)dt + K_{C} * \bar{\rho}(\bar{Z}_{t}^{i})dt \\ + \sigma_{C}\varphi_{\text{sc}} \left(|2(X_{t}^{i,N} - \bar{X}_{t}^{i}) - (C_{t}^{i,N} - \bar{C}_{t}^{i})| \right) dB_{t}^{i,sc,C} \\ - \sigma_{C}\varphi_{\text{rc}} \left(|2(X_{t}^{i,N} - \bar{X}_{t}^{i}) - (C_{t}^{i,N} - \bar{C}_{t}^{i})| \right) dB_{t}^{i,rc,C} \end{cases}$$

$$\bar{\rho} = \mathcal{L}((\bar{X}_{t}^{1}, \bar{C}_{t}^{1})),$$
(B.2)

and for $\delta > 0$, the following modified distance

$$r_t^i = \delta |X_t^{i,N} - \bar{X}_t^i| + |2(X_t^{i,N} - \bar{X}_t^i) - (C_t^{i,N} - \bar{C}_t^i)|.$$

Like previously, we consider a modified semi-metric of the form $\frac{1}{N}\sum f(r_t^i)G_t^i$ and similar calculations yield

$$d(e^{ct}f(r_t^i)G_t^i) \leq e^{ct}K_t^i dt + dM_t^i$$

where M_t^i is a continuous local martingale and

$$K_t^i = \tilde{K}_t^i + I_t^{1,i} + I_t^{2,i} + I_t^{3,i}.$$

We define \tilde{K}^i_t , $I^{1,i}_t$, $I^{2,i}_t$ and $I^{3,i}_t$ as follows

$$\begin{split} \tilde{K}_{t}^{i} = & G_{t}^{i} \left[2cf(r_{t}^{i}) + 2f''(r_{t}^{i})\sigma_{C}^{2}\varphi_{\text{rc}}\left(|2(X_{t}^{i,N} - \bar{X}_{t}^{i}) - (C_{t}^{i,N} - \bar{C}_{t}^{i})|\right)^{2} \\ & + f'(r_{t}^{i}) \left(\left|2(X_{t}^{i,N} - \bar{X}_{t}^{i}) - (C_{t}^{i,N} - \bar{C}_{t}^{i}) \left| (\delta + 1) - |(X_{t}^{i,N})^{3} - (\bar{X}_{t}^{i})^{3}|(\delta - 2) \right. \right. \\ & + |X_{t}^{i,N} - \bar{X}_{t}^{i}| \left(-\delta + \gamma + L_{X}(\delta + 2) + L_{C} \right) + |C_{t}^{i,N} - \bar{C}_{t}^{i}|(L_{X}(\delta + 2) + L_{C}) \\ & + \sigma_{C}^{2}\varphi_{\text{rc}} \left(\left|2(X_{t}^{i,N} - \bar{X}_{t}^{i}) - (C_{t}^{i,N} - \bar{C}_{t}^{i})|\right)^{2} \left(\epsilon C_{f,1} + C_{f,2}\right)r_{t}^{i} \right) \right] \\ & + \epsilon f(r_{t}^{i}) \left(4\tilde{B} - \frac{\lambda}{8}\tilde{H}(\bar{Z}_{t}^{i}) - \frac{\lambda}{8}\tilde{H}(Z_{t}^{i,N}) - \frac{\lambda}{8N} \sum_{j=1}^{N} \tilde{H}(\bar{Z}_{t}^{j}) - \frac{\lambda}{32N} \sum_{j=1}^{N} \tilde{H}(Z_{t}^{j,N}) \right), \\ I_{t}^{1,i} = & G_{t}^{i}f'(r_{t}^{i}) \left[\left(\delta + 2\right) \left(\frac{1}{N} \sum_{j=1}^{N} K_{X}(\bar{Z}_{t}^{i} - \bar{Z}_{t}^{j}) - K_{X} * \bar{\mu}_{t}(\bar{Z}_{t}^{i}) \right| \right) \\ & + \left(\frac{1}{N} \sum_{j=1}^{N} K_{C}(\bar{Z}_{t}^{i} - \bar{Z}_{t}^{j}) - K_{C} * \bar{\mu}_{t}(\bar{Z}_{t}^{i}) \right) \right], \\ I_{t}^{2,i} = & G_{t}^{i}f'(r_{t}^{i}) \left[\left(\delta + 2\right) \left(\frac{L_{X}}{N} \left(\sum_{j=1}^{N} |X_{t}^{j,N} - \bar{X}_{t}^{j}| + |C_{t}^{j,N} - \bar{C}_{t}^{j}| \right) \right) \\ & + \left(\frac{L_{C}}{N} \left(\sum_{j=1}^{N} |X_{t}^{j,N} - \bar{X}_{t}^{j}| + |C_{t}^{j,N} - \bar{C}_{t}^{j}| \right) \right) \right] \\ & - cf(r_{t}^{i})G_{t}^{i} - \epsilon f(r_{t}^{i}) \left[\frac{\lambda}{32}H(\bar{Z}_{t}^{i}) \exp\left(a\sqrt{H(\bar{Z}_{t}^{i})}\right) + \frac{\lambda}{32}H(Z_{t}^{i,N}) \exp\left(a\sqrt{H(Z_{t}^{i,N})}\right) \right] \\ & + \frac{\lambda}{32N} \sum_{j=1}^{N} H(Z_{t}^{j}) \exp\left(a\sqrt{H(Z_{t}^{j,N})}\right) \right], \end{aligned}$$

$$\begin{split} I_t^{3,i} = & \epsilon f(r_t^i) \left((\alpha_X L_X + \beta_X L_C) \left(\frac{\sum_{j=1}^N |X_t^{j,N}|}{N} \right)^2 \exp\left(a \sqrt{H(Z_t^{i,N})} \right) \\ & + (\alpha_C L_X + \beta_C L_C) \left(\frac{\sum_{j=1}^N |C_t^{j,N}|}{N} \right)^2 \exp\left(a \sqrt{H(Z_t^{i,N})} \right) \\ & - \frac{\lambda}{16} H(Z_t^{i,N}) \exp\left(a \sqrt{H(Z_t^{i,N})} \right) - \frac{\lambda}{16N} \sum_{j=1}^N H(Z_t^{j,N}) \exp\left(a \sqrt{H(Z_t^{j,N})} \right) \right). \end{split}$$

We then have the additional constraint of $\delta>2$ (so that the coefficient appearing in front of $|(X_t^{i,N})^3-(\bar{X}_t^i)^3|$ in the expression of \tilde{K}_t^i is non-positive). Otherwise, we deal with the various terms exactly as previously, through the choice of a sufficiently concave

function f and a law of large numbers, and by considering the regions of space

$$\begin{split} & \operatorname{Reg}_1^i = \left\{ (\bar{Z}_t^i, Z_t^{i,N}) \text{ s.t. } |2(X_t^{i,N} - \bar{X}_t^i) - (C_t^{i,N} - \bar{C}_t^i)| \geq \xi \text{ and } r_t^i \leq R \right\}, \\ & \operatorname{Reg}_2^i = \left\{ (\bar{Z}_t^i, Z_t^{i,N}) \text{ s.t. } |2(X_t^{i,N} - \bar{X}_t^i) - (C_t^{i,N} - \bar{C}_t^i)| < \xi \text{ and } r_t^i \leq R_1 \right\}, \\ & \operatorname{Reg}_3^i = \left\{ (\bar{Z}_t^i, Z_t^{i,N}) \text{ s.t. } r_t^i > R \right\}. \end{split}$$

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Throughout this article, we define many parameters and constants. For the sake of clarity, we list the main ones here so as to give the reader an index to refer to.

- X, C, Z: X and C are the processes we consider (see (1.1) and (1.2)) and we often refer to Z = (X, C),
- $\bar{\mu}_t = \text{Law}(\bar{\mathbf{Z}}_t)$: the density of the non-linear limit (see (1.2)),
- $\alpha, \beta, \gamma, \sigma_{\mathbf{X}}, \sigma_{\mathbf{C}}$: parameters of the problem (see (1.1)),
- $\mathbf{K_X}, \mathbf{K_C}, \mathbf{L_X}, \mathbf{L_C}, \mathbf{L_{X,max}}, \mathbf{L_{C,max}}$: K_X (resp. K_C) is a Lipschitz continuous interaction kernel, with Lipschitz constant $L_X \in [0, L_{X,max}]$ (resp. $L_C \in [0, L_{C,max}]$), as given in Assumption 1.1. In the case of uniform in time propagation of chaos, the inequalities L_X and L_C must satisfy are listed in Subsection 2.4,
- W_p : the usual Wasserstein distance associated to the L^p distance (see (1.3)),
- $\mathbf{a}, \tilde{\mathbf{a}}, \mathcal{C}_{\mathbf{init}, \mathbf{exp}}$: constants used to give an exponential initial moment to the problem (see the assumptions of Theorem 1.4 and Section 2.3),
- λ , B, \tilde{B} , H, \tilde{H} , α_X , α_C , β_X , β_C : H (resp. \tilde{H}) is a Lyapunov functions given in (2.1) (resp. (2.8)). Its main property involves parameters λ and B (resp. λ and \tilde{B}), as can for instance be seen in (2.3) (resp. (2.14)). α_X , α_C , β_X and β_C are intermediate constants given in Lemma 2.2,
- c: a contraction rate (see Subsection 2.4),
- $\mathbf{r}, \mathbf{f}, \mathbf{g}, \phi, \Phi, \mathbf{G}, \rho, \delta, \mathbf{R}, \epsilon, \mathcal{C}_{\mathbf{f},1}, \mathcal{C}_{\mathbf{f},2}$: f (see (2.20)) is a concave function, the definition of which involves g, ϕ , Φ (see Subsection 2.4). Function G (see (2.26)) is then used to define ρ (see (2.25)), the semi-metric we consider in the end. All these notations thus refer to the modified distance we consider. These functions will be applied to a modification r of the usual L^1 distance (see equation (2.24)). Then, parameters δ , R, ϵ , $\mathcal{C}_{f,1}$, and $\mathcal{C}_{f,2}$ are used to define such functions (see Subsection 2.4 for some explicit values),
- R_0, ϕ_{\min} : intermediate constants (see Subsection 2.4),
- $C_{init,2}$: uniform in time bound on the second moment of the processes (see Lemma 2.4),
- C_1, C_2, C_z : constants used to quantify the control our modified distance has over the usual L^1 and L^2 distance (see Lemma 2.7 for the control and Subsection 2.4 for explicit values),
- $\phi_{rc}, \phi_{sc}, \xi$: ϕ_{rc} and ϕ_{sc} are two Lipschitz continuous functions used to define the coupling method, and their definitions involve a parameter ξ which converges to 0 in the end (see the beginning of Section 3),
- $C_{r,H}$: used to explicit the control of the Lyapunov function H over the distance r (see Lemma 2.1),

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