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Asymptotic deviation bounds for cumulative processes

Patrick Cattiaux*, Laetitia Colombani, Manon Costa

Institut de Mathématiques de Toulouse, CNRS UMR 5219, Université Paul Sabatier, 118 route de Narbonne, F-31062 Toulouse cedex 09, France

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Abstract

The aim of this paper is to get asymptotic deviation bounds via a Large Deviation Principle (LDP) for cumulative processes also known as compound renewal processes or renewal-reward processes. These processes cumulate independent random variables occurring in time interval given by a renewal process. Our result extends the one obtained in Lefevere et al. (2011) in the sense that we impose no specific dependency between the cumulated random variables and the renewal process and the proof uses Mariani and Zambotti (2014). In the companion paper Cattiaux et al. (2022) we apply this principle to Hawkes processes with inhibition. Under some assumptions Hawkes processes are indeed cumulative processes, but they do not enter the framework of Lefevere et al. (2011).

MSC: 60F10; 60K15

Keywords: Cumulative processes; Large deviation; Deviation inequalities; Hawkes processes

1. Introduction

1.1. Cumulative processes

Cumulative processes have been introduced by Smith [14] and are applied in many purposes, such as finance where they are called *compound-renewal* processes or *renewal-reward* processes. Indeed these continuous time processes cumulate independent random variables occurring in time interval given by a renewal process. To be more specific a real valued process $(Z_t)_{t\geq 0}$ is called *a cumulative process* if the following properties are satisfied:

* Corresponding author.
E-mail addresses: patrick.cattiaux@math.univ-toulouse.fr (P. Cattiaux), laetitia.colombani@math.univ-toulouse.fr (L. Colombani), manon.costa@math.univ-toulouse.fr (M. Costa).

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(1) $Z_0 = 0$,
(2) there exists a renewal process $(S_i)_{i\geq 0}$ such that for any i, $(Z_{S_i+t} - Z_{S_i})_{t\geq 0}$ is independent of $S_0, \dots, S_i$ and $(Z_s)_{s<S_i}$,
(3) the distribution of $(Z_{S_i+t} - Z_{S_i})_{t\geq 0}$ is independent of i.

To study such processes, we write for all $t \geq 0$

$$Z_t = W_0(t) + W_1 + \cdots + W_{M_t} + r_t,$$

where $W_0(t) = Z_{t\wedge S_0}$, $(W_i)_{i\geq 1}$ are i.i.d. random variables defined by $W_i = Z_{S_i} - Z_{S_{i-1}}$, and r_t is the remaining part $r_t = Z_t - Z_{M_t}$ where M_t is the integer defined by

$$M_t = \sup\{i \geq 0, S_i \leq t\}.$$

We denote by $(\tau_i)_{i\geq 1}$ the waiting times associated to the renewal process $\tau_i = S_i - S_{i-1}$. It is worth noticing that τ_i and W_i can be dependent.

In the sequel we suppress the subscript i when dealing with the distribution (and all associated quantities like expectation, variance ...) of (τ_i, W_i) and simply use (τ, W).

A simple example of cumulative process is $Z_t = \int_0^t f(X_s)ds$ where $(X_t)_{t\geq 0}$ is a regenerative process with i.i.d. cycles [10]. Markov additive processes are other classical examples of cumulative process. In [7] the authors exhibited a renewal structure for some Hawkes processes. This description is extensively used in our companion paper [6] in order to describe such processes as cumulative processes, and to study their asymptotic behaviour.

For $\mathbb{R}$-valued cumulative processes, the law of large numbers (assuming that $\mathbb{E}[|W|]$ and $\mathbb{E}[\tau]$ are not infinite)

$$\frac{Z_t}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]} \text{ if and only if } \mathbb{E}\left(\max_{S_0\leq t<S_1} |r_t|\right) < \infty,$$

and the central limit theorem (assuming $\mathrm{Var}(W) < \infty$ and $\mathrm{Var}(\tau) < \infty$)

$$\frac{\left(Z_t - t\frac{\mathbb{E}[W]}{\mathbb{E}[\tau]}\right)}{\sqrt{t}} \xrightarrow[t\to\infty]{} \mathcal{N}\left(0, \sigma^2\right) \text{ where } \sigma^2 = \frac{1}{\mathbb{E}(\tau)}\mathrm{Var}\left(W - \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]}\tau\right)$$

can be found in Asmussen [1], theorem 3.1 and theorem 3.2.

Brown and Ross [5] have proved an equivalent of Blackwell's theorem and of the key renewal theorem for a subclass of cumulative processes, since cumulative processes are a generalization of renewal processes. Glynn and Whitt have focused in [10] on cumulative processes associated to a regenerative process and have proved law of large numbers (strong and weak), law of the iterated logarithm, central limit theorem and functional generalizations of these properties.

The aim of this work is to obtain asymptotic bounds in order to build confidence intervals. To this end we are looking at a large deviation principle (LDP) for cumulative processes. Some works have already been done. For instance, Duffy and Metcalfe [9] have considered the estimation of a rate function for a cumulative process (if it admits a LDP).

In a series of papers, Borovkov and Mogulskii [2–4] have studied the LDP (they use the term compound-renewal process), under some Cramer type assumptions. Actually, some points in their approach are not clear for us. After the submission of the present paper, Zamparo posted on ArXiv a preprint, now published in [17], that extends Borovkov–Mogulskii approach, and is based on Cramer's theory. The same author had previously studied in [16] the case of a discrete valued τ.

Another possible approach based on a higher level LDP, namely at the level of empirical measures, was developed by Lefevere, Mariani and Zambotti [12]. In this work they study specific cumulative processes where $W_i = F(\tau_i)$ for some deterministic function F which is assumed to be non-negative, bounded and continuous. In a first version of this paper, we have extended their method to general pairs (τ, W) in $\mathbb{R}^+ \times \mathbb{R}$. As suggested by the referee, our intricate proof can be simplified by using the Sanov type theorem obtained by Mariani and Zambotti in [13], what we shall do in the present work. Actually the proofs in [13] greatly simplify and extend the corresponding result for the empirical measure in [12] (as well as our previous proof of this result).

In this paper, we look at a LDP for Z_t/t in the case $r_t = 0$ and $S_0 = 0$. This assumption can be relaxed if r_t/t tends to 0 quickly enough, as it will be the case for the application to Hawkes process (see [6]), we shall briefly recall. For example, if for all $\delta > 0$

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\frac{|r_t|}{t} > \delta\right) = -\infty,$$

then Z_t/t and $(Z_t - r_t)/t$ are exponentially equivalent. They then admit the same asymptotic deviation bounds.

1.2. Motivation: Application to Hawkes processes

A Hawkes process is a point process on the real line $\mathbb{R}$ characterized by its intensity process $t \mapsto \Lambda(t)$. We consider an appropriate filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual assumptions.

Definition 1.1. Let $\lambda > 0$ and $h : (0, +\infty) \to \mathbb{R}$ a signed measurable function. Let N^0 a locally finite point process on $(-\infty, 0]$ with law $\mathbf{m}$.

The point process N^h on $\mathbb{R}$ is a Hawkes process on $(0, +\infty)$, with initial condition N^0 and reproduction measure $\mu(dt) = h(t)dt$ if:

- $N^h \mid_{(-\infty,0]} = N^0$,
- the conditional intensity measure of $N^h \mid_{(0,+\infty)}$ with respect to $(\mathcal{F}_t)_{t\geq 0}$ is absolutely continuous w.r.t the Lebesgue measure and has density:

$$\Lambda^h : t \in (0, +\infty) \mapsto f\left(\lambda + \int_{(-\infty,t)} h(t-u) N^h(du)\right). \tag{1.1}$$

for some non-negative function f.

Hawkes processes have been introduced by Hawkes [11]. Most of the literature concerned with the large time behaviour of $N_t^h = N^h([0, t])$ is dedicated to the case $h \geq 0$ (self excitation). This behaviour is studied in detail in [6] when h is a signed (the negative part modelling self inhibition) compactly supported function, and the function f (called the jump rate function) is given by

$$f(u) = \max(0, u).$$

In this situation one gets a description of N_t^h as a cumulative process (see [6] subsection 2.3) with few information on the joint law of (τ, W). This was the initial motivation for the present work. In particular, controlling the asymptotic deviation from the mean, in this framework with unbounded W_i's, can lead to asymptotic confidence intervals. We refer to Corollary 2.13 [6] for a more complete overview and explicit results in this situation. We shall discuss this situation later.

2. Notations and main result

2.1. First notations

We consider $(\tau_i, W_i)_{i\geq 1}$ an i.i.d. sequence of pairs of random variables built on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $[0, +\infty] \times \mathbb{R}$. Actually we are mainly interested in the case where W takes non-negative values which is the case for Hawkes processes.

The law of (τ_i, W_i) is an arbitrary probability measure ψ on $(0, +\infty) \times \mathbb{R}$. We denote this by: $(\tau_i, W_i) \sim \psi$. In the sequel we generically use the notation (τ, W) for a pair with the same distribution as (τ_i, W_i). Notice that we thus assume that

$$\psi(\tau = 0) = \psi(\tau = +\infty) = 0$$

which is Assumption (A1) in [13], implying in particular that $\mathbb{E}(\tau) > 0$.

We denote by $\mathcal{M}^1(\mathcal{X})$ the space of probability measure on some measurable space $(\mathcal{X}, \mathcal{G})$.

We consider the renewal process associated with $(\tau_i)_{i\geq 1}$:

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} \tau_i,$$
$$M_t = \sup \{n \geq 0, S_n \leq t\} .$$

We will study the quantity:

$$Z_t = \sum_{i=1}^{M_t} W_i, \tag{2.1}$$

where as usual an empty sum is equal to 0.

The first main goal of this paper is to prove a Large Deviation Principle for the process $(Z_t/t)_{t\geq 0}$. Let us recall some basic definitions in large deviation theory (we refer to [8]).

A family of probability measures $(\eta_t)_{t\geq 0}$ on a topological space $(\mathcal{X}, T_{\mathcal{X}})$ equipped with its Borel σ-field, satisfies the Large Deviations Principle (LDP) with rate function $J(.)$ and speed $\gamma(t) = t$ if J is lower semi-continuous from $\mathcal{X}$ to $[0, +\infty]$, and the following holds

$$-\inf_{x\in\mathcal{O}} J(x) \leq \liminf_{t\to+\infty} \frac{1}{t} \log \eta_t(\mathcal{O}) \quad \text{for all open subset } \mathcal{O}, \tag{2.2}$$

and

$$-\inf_{x\in\mathcal{C}} J(x) \geq \limsup_{t\to+\infty} \frac{1}{t} \log \eta_t(\mathcal{C}) \quad \text{for all closed subset } \mathcal{C}. \tag{2.3}$$

We shall say that $(\eta_t)_{t\geq 0}$ satisfies the *full* LDP when (2.2) and (2.3) are satisfied, while we will use *weak* LDP when $\mathcal{C}$ closed is replaced by $\mathcal{C}$ compact in (2.3). When η_t is the distribution of some random variable Y_t (for instance Z_t/t) we shall say that the family $(Y_t)_{t\geq 0}$ satisfies a LDP.

Since J is lower semi-continuous the level sets $\{x \in \mathcal{X}, J(x) \leq a\}$ are closed. If in addition they are compact, then J is said to be a good rate function.

In this paper we only consider the speed function $\gamma(t) = t$ so that we will no more refer to it.

A particularly important notion for our purpose is the notion of *exponentially good approximation*.

Definition 2.1. Assume that $(\mathcal{X}, d)$ is a metric space. A family of random variables $\{(Y_{n,t})_{t\geq 0}\}_{n\in\mathbb{N}}$ is an exponentially good approximation of $(Y_t)_{t\geq 0}$ (all these variables being defined on the same probability space $(\Omega, \mathbb{P})$), if for all $\delta > 0$ it holds

$$\lim_{n\to\infty} \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(d(Y_{n,t}, Y_t) > \delta) = -\infty .$$

The key result is then

Theorem 2.2. *In the framework of Definition 2.1, assume that $\{(Y_{n,t})_{t\geq 0}\}_{n\in\mathbb{N}}$ is an exponentially good approximation of $(Y_t)_{t\geq 0}$. Then the following statements hold true.*

(1) If $\{(Y_{n,t})_{t\geq 0}\}_{n\in\mathbb{N}}$ satisfies a full LDP with rate function J^n then $(Y_t)_{t\geq 0}$ satisfies a weak LDP with rate function

$$J(x) = \sup_{\delta>0} \liminf_{n\to\infty} \inf_{d(y,x)<\delta} J^n(y) .$$

(2) If $\mathcal{X}$ is locally compact, then the same conclusion is true when $\{(Y_{n,t})_{t\geq 0}\}_{n\in\mathbb{N}}$ satisfies only a weak LDP.

(3) If J (defined above) is a good rate function such that for any closed set F,

$$\inf_{y\in F} J(y) \leq \limsup_{n\to\infty} \inf_{y\in F} J^n(y),$$

then $(Y_t)_{t\geq 0}$ satisfies a full LDP with rate function J.

The first and last points in the previous Theorem are contained in [8] Theorem 4.2.16. The second one is a consequence of the fact that closed balls are compact sets. Usually, the Theorem is sufficient to prove a full LDP. Nevertheless, in some cases, the study of the rate function J is difficult. The lemma below gives an alternative, using exponential tightness which is easy to obtain with our assumptions.

Lemma 2.3. *If $(Y_t)_{t\geq 0}$ satisfies a weak LDP with a rate function I and is exponentially tight, i.e. for all $\alpha > 0$, there exists a compact set K_α such that*

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(Y_t \notin K_\alpha^c\right) < -\alpha,$$

then $(Y_t)_{t\geq 0}$ satisfies a full LDP and I is a good rate function.

This Lemma is a consequence of the Lemma 1.2.18 in [8].

2.2. Main results

Introduce the following quantities

$$\theta_0 := \sup_{\theta\geq 0} \{ \mathbb{E}[\mathrm{e}^{\theta\tau}] < \infty \} , \tag{2.4}$$

and

$$\eta_0 := \sup_{\eta\geq 0} \{ \mathbb{E}[\mathrm{e}^{\eta|W|}] < \infty \} . \tag{2.5}$$

Also introduce the classical Cramer transform, for $(a, b) \in \mathbb{R}^2$,

$$\Lambda^*(a, b) = \sup_{(x,y)\in\mathbb{R}^2} \{ax + by - \log \mathbb{E}(e^{x\tau + yW})\} . \tag{2.6}$$

We finally define, for $(m, \beta, x, y) \in \mathbb{R}^4$,

$$\Lambda(m, \beta, x, y) = x + my - \beta \log \mathbb{E}\left(e^{x\tau + yW}\right) \tag{2.7}$$

and the rate function J for any $m \in \mathbb{R}$,

$$\begin{aligned} J(m) &= \inf_{\beta>0} \beta \Lambda^*\left(\frac{1}{\beta}, \frac{m}{\beta}\right), \\ &= \inf_{\beta>0} \sup_{x,y} \Lambda(m, \beta, x, y) . \end{aligned} \tag{2.8}$$

We then may state

Theorem 2.4. *Assume that $\eta_0 > 0$ and $\theta_0 > 0$. Let J be given by (2.8) and $\bar{J}$ be defined as*

$$\bar{J}(m) = J(m) \text{ for } m \neq 0,$$
$$\bar{J}(0) = \min(J(0), \theta_0) .$$

- *If $\eta_0 = +\infty$ (in particular if W is bounded) then $(Z_t/t)_{t\geq 0}$ satisfies a full LDP with good rate function $\bar{J}$.*
- *If $\eta_0 < +\infty$, denoting $m = \mathbb{E}(W)/\mathbb{E}(\tau)$ we have for all $a > 0$ and $\kappa \in (0, 1)$*

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\frac{Z_t}{t} \geq m + a\right) \leq - \min\left[\inf_{z\geq m+\kappa a} \bar{J}(z) , \frac{\eta_0 a(1-\kappa)}{4}\right], \tag{2.9}$$

and similarly

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\frac{Z_t}{t} \leq m - a\right) \leq - \min\left[\inf_{z\leq m-\kappa a} \bar{J}(z) , \frac{\eta_0 a(1-\kappa)}{4}\right]. \tag{2.10}$$

Remark 2.5. A short discussion. As we said in [6], the direct Cramer's approach in e.g. [17] furnishes more general results but with a much less explicit rate function.

In particular, contrary to [17], when $\eta_0 < +\infty$ we do not provide a LDP principle but asymptotic deviation bounds. These bounds are actually what is useful from a statistical point of view, since they allow to build confidence intervals around the asymptotic mean.

Due to the fact that we are using the results in [13], the method we will develop here extends immediately to W taking its values in $\mathbb{R}^k$ or even in a general infinite dimensional normed vector space, provided $\theta_0 = +\infty$ in the latter case. Actually, most of the work in the present paper is about understanding the rate function, and giving a tractable form for it. ◇

3. Large deviations for the empirical measure

Following [13], we introduce the empirical measure

$$\mu_t := \frac{1}{t} \int_{[0,t)} \delta_{(\tau_{M_s+1}, W_{M_s+1})} \, ds , \tag{3.1}$$

so that, considering

$$\varphi(u, w) = \frac{w}{u}$$

one has

$$\mu_t(\varphi) := \int \varphi \, d\mu_t = \frac{1}{t} \int_0^t \frac{W_{M_s+1}}{\tau_{M_s+1}} ds$$

$$= \frac{1}{t}\sum_{i=1}^{M_t}\int_{S_{i-1}}^{S_i} \frac{W_i}{\tau_i} ds + \frac{1}{t}\int_{S_{M_t}}^{t} \frac{W_{M_t+1}}{\tau_{M_t+1}} ds$$
$$= \frac{Z_t}{t} + \frac{t-S_{M_t}}{t}\,\frac{W_{M_t+1}}{\tau_{M_t+1}}\,, \tag{3.2}$$

if the latter makes sense.

We will thus deduce a LDP for $(Z_t/t)_{t\geq 0}$ from a LDP for $(\mu_t)_{t\geq 0}$ and the contraction principle ([8] Theorem 4.2.1). The LDP for $(\mu_t)_{t\geq 0}$ is precisely the aim of the work by Mariani and Zambotti [13]. We have to introduce some more notations.

First, for the sake of simplicity we still assume that $\mathcal{X} = (0,+\infty)\times\mathbb{R}$ so that Assumption (A4) (i.e. $\mathcal{X}$ locally compact) in [13] is satisfied. The generic point in $\mathcal{X}$ is denoted by $x = (u,w)$. The application denoted by τ in [13] is thus simply $(u,w)\mapsto u$ in our setting.

This immediately implies that Assumption (A2) in [13] is satisfied, since for all $x = (u,w)\in (0,+\infty)\times\mathbb{R}$ it holds

$$\zeta(x) = \inf_{\delta>0}\sup\left\{c\geq 0 \,:\, \int_{B((u,w),\delta)} e^{cu'}\,\psi(du',dw') < +\infty\right\} = +\infty\,.$$

Assumption (A3) therein is equivalent to $\theta_0 = +\infty$ and we shall not use it.

The set of non-negative Radon measures on $\mathcal{X}$ with total mass less than or equal to 1 is denoted by $\bar{\mathcal{M}}^1(\mathcal{X})$. The main advantage of considering this set is that it is compact and Polish for the vague topology i.e. the weakest topology such that for any continuous and compactly supported f, the map $\nu\mapsto\int f\,d\nu := \nu(f)$ is continuous. Recall that if f is continuous, bounded and goes to 0 at infinity (i.e. $\sup_{|x|>R}|f(x)|\to 0$ as $R\to\infty$), then the application $\nu\mapsto\nu(f)$ is continuous on $\bar{\mathcal{M}}^1(\mathcal{X})$.

We denote by $\mathcal{M}^1(\mathcal{X})$ the set of probability measures on $\mathcal{X}$. In [12] to $\nu\in\bar{\mathcal{M}}^1(\mathcal{X})$ is associated the probability measure

$$\widetilde{\nu}(dx) = \nu(dx) + (1-\nu(\mathcal{X}))\delta_\partial$$

where $\mathcal{X}\cup\partial$ denotes the one point compactification of $\mathcal{X}$.

In both papers the authors then introduce, provided $0 < \nu(1/u) := \int \frac{1}{u}\,\nu(du,dw) < +\infty$,

$$\bar{\nu}(dx) = \bar{\nu}(du,dw) := \frac{1}{\nu(1/u)}\,\frac{1}{u}\,\nu(du,dw)\,. \tag{3.3}$$

Finally recall that if π and π' are probability measures on $\mathcal{X}$, the relative entropy of π w.r.t. π' is defined as

$$H(\pi|\pi') = \begin{cases}\int \log\left(\dfrac{d\pi}{d\pi'}\right)d\pi & \text{if } \pi \text{ is absolutely continuous w.r.t. } \pi'\\ +\infty & \text{otherwise.}\end{cases}$$

Since Assumptions (A1), (A2) and (A4) are satisfied, Proposition 1.5 and Theorem 1.6 in [13] then imply in our framework

Theorem 3.1. *Define* $I : \bar{\mathcal{M}}^1(\mathcal{X}) \to [0,+\infty]$ *as*

$$I(\nu) = \begin{cases}\nu(1/u)H(\bar{\nu}|\psi) + (1-\nu(\mathcal{X}))\theta_0\,, & \textit{if } 0<\nu(1/u)<+\infty\\ \theta_0\,, \quad \textit{if } \nu \textit{ is the null measure}\\ +\infty, \textit{ otherwise.}\end{cases} \tag{3.4}$$

Then I is convex, is a good rate function and the family $(P_t)_{t\geq 0}$ of the probability distributions of $(\mu_t)_{t\geq 0}$ satisfies a full LDP with rate function I and speed t.

The specific case where ν is the null measure will play a special role. Notice that under our hypotheses the null measure is the only one such that $\nu(1/u) = 0$.

Moreover if ν has a singular part denoted by ν_s, then $\nu_s(\zeta) = +\infty$ since $\zeta(x) = +\infty$ for all x. Therefore our definition of $I(\nu)$ matches the one in [13].

An immediate corollary can then be obtained using the contraction principle in a specific case.

Corollary 3.2. *Assume in addition that there exists positive constants K and $\varepsilon < 1$ such that $\psi(|w| \leq K$ and $u \geq \varepsilon) = 1$. Then, $(\mu_t(\varphi))_{t\geq 0}$ satisfies a full LDP with the convex good rate function*

$$\bar{J}(m) = \inf\{I(\nu)\,,\ \nu \in \bar{\mathcal{M}}^1(\mathcal{X}) :\ \nu(\varphi) = m\}\,, \tag{3.5}$$

where as usual the infimum on an empty set is $+\infty$.

Proof. Let η_K be a continuous function such that $\mathbf{1}_{|w|\leq K} \leq \eta_K(w) \leq \mathbf{1}_{|w|\leq 2K}$. Introduce

$$\varphi_{K,\varepsilon}(u, w) = \frac{w}{u \vee \varepsilon}\,\eta_K(w)\,.$$

First remark that under our assumptions on ψ, $\mu_t(\varphi) = \mu_t(\varphi_{K,\varepsilon})$ almost surely. Since $\varphi_{K,\varepsilon}$ is continuous, bounded and goes to 0 at infinity, $\nu \mapsto \nu(\varphi_{K,\varepsilon})$ is continuous. One can thus apply the contraction principle, yielding a full LDP with good rate function

$$\bar{J}_{K,\varepsilon}(m) = \inf\{I(\nu)\,,\ \nu \in \bar{\mathcal{M}}^1(\mathcal{X}) :\ \nu(\varphi_{K,\varepsilon}) = m\}\,.$$

If one of $\bar{J}$ or $\bar{J}_{K,\varepsilon}$ is finite then ν is necessarily absolutely continuous w.r.t. ψ (including the case of the null measure) so that $|W| \leq K$ and $\tau \geq \varepsilon$, ν almost everywhere. Accordingly $\nu(\varphi) = \nu(\varphi_{K,\varepsilon})$ and $\bar{J} = \bar{J}_{K,\varepsilon}$. □

To obtain our main result, it remains to relax the boundedness assumptions on τ and W and to compare $\bar{J}$ and J defined in (2.8) and (3.5). The next result is a first step in this direction, removing the assumption on τ.

Proposition 3.3. *Assume that there exists a positive constant K such that $\psi(|w| \leq K) = 1$. Then for $m \neq 0$, $\bar{J}(m) = J(m)$ while for $m = 0$, $\bar{J}(0) = \min(J(0), \theta_0)$, where J is defined in (2.8).*

Proof. The proof is inspired by the proof of Lemma 5.1 in [12].

First remark that if $\nu \in \bar{\mathcal{M}}^1(\mathcal{X})$, introducing the normalized $\nu_1 = \nu/\nu(\mathcal{X})$ (except if $\nu = 0$), one has on the one hand $\bar{\nu_1} = \bar{\nu}$ and on the other hand

$$I(\nu) = \nu(\mathcal{X})\,\nu_1(1/u)\,H(\bar{\nu_1}|\psi) + (1 - \nu(\mathcal{X}))\,\theta_0\,,$$

provided $\nu(1/u) < +\infty$.

Since for a non null ν, $\nu(\mathcal{X})$ can be any $\alpha \in]0, 1]$, we deduce that, defining

$$\bar{J}_1(m) = \inf\{\alpha\ \nu_1(1/u)\,H(\bar{\nu}_1|\psi) + (1-\alpha)\theta_0\,;$$
$$\alpha \in]0, 1], \nu_1 \in \mathcal{M}^1(\mathcal{X}), \nu_1(1/u) < +\infty, \nu_1(\varphi) = \frac{m}{\alpha}\Big\}\,,$$

one has

$$\bar{J}(m) = \bar{J}_1(m) \text{ for } m \neq 0 \quad ; \quad \bar{J}(0) = \min(\bar{J}_1(0), \theta_0)\,,$$

since for $m = 0$ one has to also consider the null measure.

Since $\bar{\nu}(w) = \nu(\varphi)/\nu(1/u)$ and $\nu(1/u) = 1/\bar{\nu}(u)$, it is elementary to see that

$$\bar{J}_1(m) = \inf_{\alpha\in]0,1]\,,\gamma>0\,,\nu'\in\mathcal{M}^1(\mathcal{X})} \{(\alpha/\gamma)\, H(\nu'|\psi) + (1-\alpha)\theta_0;\, \nu'(u) = \gamma,\, \nu'(w) = \gamma\, m/\alpha\}\,, \tag{3.6}$$

the correspondence being $\nu' = \bar{\nu}_1$ i.e $\nu_1 = (1/\nu'(u))\, u\, \nu'$.

Now we can mimic what is done in [12].

Let $p(a, b) = \inf\{H(\nu'|\psi)\,;\, \nu' \in \mathcal{M}^1(\mathcal{X}),\, \nu'(u) = a,\, \nu'(w) = b\}$. We have

$$\begin{aligned} p^*(x, y) &= \sup_{a,b\in\mathbb{R}^2} (ax + by - p(a, b)) \\ &= \sup_{a,b\in\mathbb{R}^2,\nu'\in\mathcal{M}^1(\mathcal{X})} \{ax + by - H(\nu'|\psi);\, \nu'(u) = a,\, \nu'(w) = b\} \\ &= \sup_{\nu'\in\mathcal{M}^1(\mathcal{X})} \{\nu'(xu + yw) - H(\nu'|\psi)\} = \log \psi(e^{x\tau+yW}) \\ &= \Lambda(x, y) \end{aligned}$$

thanks to the variational definition of the relative entropy. Since p is lower semi continuous and convex we have $p = (p^*)^* = \Lambda^*$.

We thus deduce that

$$\bar{J}_1(m) = \inf\left\{\frac{\alpha}{\gamma}\Lambda^*\left(\gamma, \frac{m\gamma}{\alpha}\right) + (1-\alpha)\theta_0\,;\, \alpha\in]0,1],\, \gamma > 0\right\}\,.$$

But

$$\frac{\alpha}{\gamma}\Lambda^*\left(\gamma, \frac{m\gamma}{\alpha}\right) = \beta\Lambda^*\left(\frac{\alpha}{\beta}, \frac{m}{\beta}\right) \text{ where } \beta = \frac{\alpha}{\gamma}.$$

Thus

$$\bar{J}_1(m) = \inf\left\{\beta\Lambda^*\left(\frac{\alpha}{\beta}, \frac{m}{\beta}\right) + (1-\alpha)\theta_0\,;\, \alpha\in]0,1],\, \beta > 0\right\}.$$

We will show that, for any $\beta > 0$

$$\inf_{\alpha\in]0,1]}\left\{\beta\Lambda^*\left(\frac{\alpha}{\beta}, \frac{m}{\beta}\right) + (1-\alpha)\theta_0\right\} = \beta\Lambda^*\left(\frac{1}{\beta}, \frac{m}{\beta}\right).$$

Taking $\alpha = 1$, we see that the left hand side is less than or equal to the right hand side. To show the converse inequality, let us pick $\alpha \in]0, 1]$:

$$\begin{aligned} \beta\Lambda^*\left(\frac{\alpha}{\beta}, \frac{m}{\beta}\right) + (1-\alpha)\theta_0 &= \sup_{x,y\in\mathbb{R}^2} \{\alpha x + (1-\alpha)\theta_0 + my - \beta\Lambda(x, y)\} \\ &\geq \sup_{x,y\in\mathbb{R}^2} \{x \wedge \theta_0 + my - \beta\Lambda(x, y)\}. \end{aligned}$$

Since W is bounded, $e^{yW} \geq C(y) > 0$ for all y, so that we have for all $x > \theta_0$ and all y,

$$\psi(\mathrm{e}^{x\tau+yW}) \geq C(y)\, \psi(\mathrm{e}^{x\tau}) = +\infty\,.$$

This shows that $\Lambda(x, y) = +\infty$, for all $x > \theta_0$ and for all y. Hence, the supremum on x can be restricted to the supremum on $\{x \leq \theta_0\}$:

$$\begin{aligned}\beta\Lambda^*\left(\frac{\alpha}{\beta}, \frac{m}{\beta}\right) + (1-\alpha)\theta_0 &\geq \sup_{x,y\in\mathbb{R}^2} \{x \wedge \theta_0 + my - \beta\Lambda(x, y)\} \\ &= \sup_{x\leq\theta_0, y\in\mathbb{R}} \{x + my - \beta\Lambda(x, y)\} \\ &= \beta\Lambda^*\left(\frac{1}{\beta}, \frac{m}{\beta}\right)\end{aligned}$$

and the desired inequality is proved. □

Remark 3.4. Let us remark on a simple example that the rate function J defined in (2.8) is not lower semi continuous. If $W = 1$, one has $Z_t = M_t$ and one easily sees that (recall (2.7)) $\sup_{x,y\in\mathbb{R}^2} \Lambda(m, \beta, x, y) = +\infty$ except for $\beta = m$ yielding $J(m) = \sup_x\{x - m \log \mathbb{E}(e^{x\tau})\}$ as expected. Notice that $J(0) = +\infty$ since $\beta > 0$. In particular if τ has an exponential distribution with parameter 1, $\theta_0 = 1$, Z_t is the standard Poisson process and $J(m) = 1 - m + m \log m$ for $m > 0$ while $J(m) = +\infty$ if $m \leq 0$. Accordingly J is not lower semi continuous at $m = 0$, and $\bar{J}$ is precisely the lower semi continuous envelope of J.

We did not check correctly this point in the previous version of the paper and the same minor mistake is made in Lemma 5.1 of [12]. ◇

One can ask about whether the infimum defining J_1 is achieved or not, hence is a minimum. This question is briefly studied in Lemma 5.1 of [12], where the argument p.22, showing that π_n therein is tight, sounds strange. Let us give a complete proof.

Proposition 3.5. *Under the assumptions of Proposition 3.3, for $m \neq 0$, the infimum in (3.5) is a minimum, provided it is finite.*

Proof. We use the expression (3.6) in order to prove the proposition. Assume that $m \neq 0$. If $\bar{J}_1(m) < +\infty$ consider a minimizing sequence $(\gamma_n, \alpha_n, \nu'_n)_{n\geq 0}$. First, $H(\nu'_n|\psi) < +\infty$ (at least for large n's), so that ν'_n is absolutely continuous with respect to ψ, and so $\nu'_n(|w| \leq K) = 1$. It follows that $\gamma_n/\alpha_n \leq K/|m|$ hence $\gamma_n \leq K/|m|$.

Since $\alpha_n \in]0, 1]$ and γ_n is bounded, one can find a subsequence still denoted $(\alpha_n, \gamma_n)_{n\geq 0}$ converging to $(\alpha, \gamma) \in [0, 1] \times [-K/|m|, K/|m|]$. In addition, for n large enough,

$$(\alpha_n/\gamma_n)H(\nu'_n|\psi) \leq \bar{J}_1(m) + 1 := C$$

so that

$$H(\nu'_n|\psi) \leq C\,(\gamma_n/\alpha_n) \leq C\,(K/|m|)\,.$$

Since the entropy is bounded, the sequence $(\nu'_n)_{n\geq 0}$ is tight and one can thus also find a subsequence weakly converging to ν'_∞ which satisfies $H(\nu'_\infty|\psi) \leq \liminf_n H(\nu'_n|\psi) < +\infty$ thanks to the lower semi continuity of the entropy w.r.t. the first variable.

Recall that $\gamma_n = \nu'_n(u)$. For all $M > 0$, we have that $\gamma_n \geq \nu'_n(u \wedge M)$ and taking the limit in n, we deduce that $\nu'_\infty(u \wedge M) = \lim_n \nu'_n(u \wedge M) \leq \gamma$ and finally using the monotone convergence $\nu'_\infty(u) = \gamma' \leq \gamma$. We deduce in particular that $\gamma > 0$ since $\nu'_\infty(u = 0) = 0$ because the measure ν'_∞ is absolutely continuous w.r.t. ψ and $\psi(u=0)=0$ by assumption. Moreover, since $K \geq \gamma_n|m|/\alpha_n$ and $\gamma_n \to_{n\to\infty} \gamma > 0$, we also have that $\alpha = \lim_{n\to\infty}\alpha_n > 0$. In addition, from the absolute continuity of ν'_n and ν'_∞ w.r.t. ψ, we deduce that $\nu'_\infty(|w| \leq K) = 1$ and

$$m\gamma/\alpha = \lim_n \nu'_n(w) = \lim_n \nu'_n(w\,\mathbf{1}_{|w|\leq K}) = \nu'_\infty(w\,\mathbf{1}_{|w|\leq K}) = \nu'_\infty(w)\,.$$

Introduce $\nu_n = (1/\gamma_n)\,u\,\nu'_n$. ν_n is a sequence of probability measures that vaguely converges to ν_∞ satisfying $\nu_\infty(\mathcal{X}) = \gamma'/\gamma$, $\nu_\infty(1/u) = 1/\gamma$ and $\nu_\infty(\varphi) = m/\alpha$. Of course $\bar{\nu}_\infty = \nu'_\infty$.

According to Lemma 2.3 and Lemma 2.2 in [12] (based on the variational formula for the entropy)

$$\liminf_n \frac{1}{\gamma_n}\, H(\nu'_n|\psi) \geq (\gamma'/\gamma)\frac{1}{\gamma} H(\nu'_\infty|\psi) + (1-(\gamma'/\gamma))\theta_0\,.$$

Finally define $\mu_\infty = \alpha\nu_\infty$ so that $\mu_\infty(\mathcal{X}) = \alpha(\gamma'/\gamma) \leq 1$ and $\mu_\infty \in \bar{\mathcal{M}}^1(\mathcal{X})$. From what precedes we deduce

$$\begin{aligned}
\bar{J}_1(m) &= \liminf_n \big((\alpha_n/\gamma_n)H(\nu'_n|\psi) + (1-\alpha_n)\theta_0\big)\\
&\geq \mu_\infty(1/u)\, H(\bar{\mu}_\infty|\psi) + ((1-\alpha) + \alpha(1-(\gamma'/\gamma))\theta_0)\\
&= \mu_\infty(1/u)\, H(\bar{\mu}_\infty|\psi) + (1-\mu_\infty(\mathcal{X}))\theta_0
\end{aligned}$$

and in addition $\mu_\infty(\varphi) = m$. Hence the infimum for $\bar{J}(m)$ is achieved at μ_∞. $\square$

4. Large deviations for the cumulative process when W is bounded

In this section, we shall deduce a LDP for $(Z_t/t)_{t\geq 0}$ starting with (3.2). We still assume that W is a bounded random variable, therefore it consists in relaxing the assumption on τ in Corollary 3.2.

To this end, for $\varepsilon > 0$, introduce $\tau^\varepsilon = \tau + \varepsilon$ and ψ^ϵ the distribution of (τ^ε, W). We then define I^ε as in (3.4), replacing ψ by ψ^ε, and $\bar{J}^\varepsilon$ as in (3.5) replacing I by I^ε.

Theorem 4.1. *Assume that there exists a positive constant K such that $\psi(|w| \leq K) = 1$. Then, $(Z_t/t)_{t\geq 0}$ satisfies a full LDP with the good convex rate function $\bar{J}$.*

Proof. The proof will be done in several steps.

Step 1. We shall first prove the

Lemma 4.2. *Assume that there exists a positive constant K such that ψ-almost surely, $|W| \leq K$. Then, $(Z_t/t)_{t\geq 0}$ satisfies a weak LDP with the convex rate function*

$$\widetilde{J}(m) = \sup_{\delta>0} \liminf_{\varepsilon\to 0} \inf_{|z-m|<\delta} \bar{J}^\varepsilon(z)\,. \tag{4.1}$$

Proof of the lemma. Following the same lines as in (3.2)

$$\mu_t^\varepsilon(\varphi) = \frac{1}{t}\sum_{i=1}^{M_t^\varepsilon} W_i + \frac{(t - S^\varepsilon_{M_t^\varepsilon})W_{M_t^\varepsilon+1}}{t\,\tau^\varepsilon_{M_t^\varepsilon+1}}\,.$$

Since $\tau^\varepsilon \geq \tau$, we deduce that $M_t^\varepsilon \leq M_t$. Accordingly

$$\left|\mu_t^\varepsilon(\varphi) - \frac{1}{t}\sum_{i=1}^{M_t} W_i\right| \leq \frac{1}{t}\left|\sum_{i=M_t^\varepsilon+1}^{M_t} W_i\right| + \left|\frac{(t-S^\varepsilon_{M_t^\varepsilon})W_{M_t^\varepsilon+1}}{t\,\tau^\varepsilon_{M_t^\varepsilon+1}}\right|$$
$$\leq \frac{K}{t}\,\big((M_t - M_t^\varepsilon)+1\big)\,.$$

Using Theorem 2.2, it is then sufficient to prove that $(M_t^\varepsilon/t)_\varepsilon$ is an exponentially good approximation of M_t/t, i.e. that

$$\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(|M_t - M_t^\varepsilon| > \delta\, t) = -\infty\,.$$

The proof is similar to the one of [12] Lemma 5.4 where a different approximation is used. Denote as usual by $\lfloor x\rfloor$ the integer part of $x\in\mathbb{R}$. Recall that $M_t^\varepsilon \leq M_t$ and $S_n^\varepsilon = S_n + n\varepsilon$. Choose some $\delta > 0$ and $A > 0$. Then

$$\begin{aligned}
\mathbb{P}(M_t - M_t^\varepsilon > t\delta) &\leq \sum_{n=1}^{\lfloor At\rfloor}\mathbb{P}(M_t - M_t^\varepsilon > t\delta\,;\ M_t = n) + \mathbb{P}(M_t > \lfloor At\rfloor)\\
&= \sum_{n=1}^{\lfloor At\rfloor}\mathbb{P}(M_t^\varepsilon < n - t\delta\,;\ M_t = n) + \mathbb{P}(S_{\lfloor At\rfloor}\leq t)\\
&\leq \sum_{n=1}^{\lfloor At\rfloor}\mathbb{P}(S^\varepsilon_{\lfloor n-t\delta\rfloor}\geq t;\ M_t = n) + \mathbb{P}(S_{\lfloor At\rfloor}\leq t)\\
&\leq \sum_{n=1}^{\lfloor At\rfloor}\mathbb{P}(S_{\lfloor n-t\delta\rfloor}\geq t - (n-t\delta)\varepsilon;\ S_n\leq t) + \mathbb{P}(S_{\lfloor At\rfloor}\leq t)\\
&\leq \sum_{n=1}^{\lfloor At\rfloor}\mathbb{P}(S_n - S_{\lfloor n-t\delta\rfloor}\leq (n-t\delta)\varepsilon) + \mathbb{P}(S_{\lfloor At\rfloor}\leq t)\\
&\leq At\,\mathbb{P}(S_{\lfloor t\delta\rfloor}\leq At\varepsilon) + \mathbb{P}(S_{\lfloor At\rfloor}\leq t)\,,
\end{aligned}$$

where we have used that the distribution of $S_j - S_k$ is the one of S_{j-k} for any positive integers $j\geq k$.

According to Markov inequality

$$\mathbb{P}(S_{\lfloor t\delta\rfloor}\leq At\varepsilon) = \mathbb{P}(e^{-S_{\lfloor t\delta\rfloor}/\varepsilon}\geq e^{-At}) \leq \exp(At + \lfloor t\delta\rfloor\,\log\mathbb{E}(e^{-\tau/\varepsilon}))\,.$$

Thus

$$\limsup_{t\to\infty}\frac{1}{t}\log(At\,\mathbb{P}(S_{\lfloor t\delta\rfloor}\leq At\varepsilon)) = A + \delta\,\log\mathbb{E}(e^{-\tau/\varepsilon})\,.$$

Since $\log\mathbb{E}(e^{-\tau/\varepsilon})\to_{\varepsilon\to 0} -\infty$, we have

$$\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log(At\,\mathbb{P}(S_{\lfloor t\delta\rfloor}\leq At\varepsilon)) = -\infty\,.$$

Similarly

$$\mathbb{P}(S_{\lfloor At\rfloor}\leq t)\leq \exp(t + \lfloor At\rfloor\,\log\mathbb{E}(e^{-\tau}))\,,$$

so that choosing A large enough, we can make $\frac{1}{t}\log\mathbb{P}(S_{\lfloor At\rfloor}\leq t)$ as small as we want i.e. less than $-B$ for any given $B>0$. It is then enough to let ε go to 0 and then B go to infinity to obtain the result. $\square$

In particular we know from Theorem 2.2 that $\widetilde{J}$ is lower semi-continuous so that its level sets are closed.

Step 2. We shall now identify $\widetilde{J}$ with $\bar{J}$. Recall that for all $m\neq 0$, $\bar{J}(m)=J(m)=\inf_{\beta>0}\sup_{x,y\in\mathbb{R}^2}\Lambda(m,\beta,x,y)$, where Λ is defined in (2.7).

Lemma 4.3. *Under the assumptions of Lemma 4.2, $\widetilde{J}\geq\bar{J}$.*

Proof of the Lemma. Since $\tau>0$ almost surely, one can find $x_\tau<0$ such that $\mathbb{E}(e^{x_\tau\,\tau})=e^{-1}$ so that

$$\sup_{x,y}\Lambda(m,\beta,x,y)\geq\sup_{x}\Lambda(m,\beta,x,0)\geq x_\tau+\beta\,.$$

In particular if $J(m)<+\infty$ the infimum in β has to be taken for $\beta\leq J(m)-x_\tau=\beta_\tau$.

From now we assume that $\widetilde{J}<+\infty$, indeed if $\widetilde{J}(m)=+\infty$, the inequality $\bar{J}(m)\leq\widetilde{J}(m)$ clearly holds. The key remark is the following equality

$$\Lambda^\varepsilon(m,\beta,x,y)=\Lambda(m,\beta,x,y)-x\beta\varepsilon\,. \tag{4.2}$$

If $\theta_0<+\infty$ it immediately follows from (4.2) and the fact that according to the proof of Proposition 3.3 the supremum in $\bar{J}$ can be restricted to $\{x\leq\theta_0\}$ that

$$\bar{J}(m)\leq\bar{J}^\varepsilon(m)+\beta_\tau\,\varepsilon\,\theta_0,$$

for the case $m=0$ just remark in addition that $\theta_0\leq\theta_0(1+\beta_\tau\varepsilon)$.

One can find a sequence $(m_n,\varepsilon_n)_{n\geq 0}$ going to $(m,0)$ such that $\widetilde{J}(m)=\liminf_{n\to\infty}\bar{J}^{\varepsilon_n}(m_n)$. Since $\bar{J}$ is lower semi continuous,

$$\bar{J}(m)\leq\liminf_{n\to\infty}\bar{J}(m_n)\leq\liminf_{n\to\infty}(\bar{J}^{\varepsilon_n}(m_n)+\theta_0\beta_\tau\,\varepsilon_n)=\widetilde{J}(m)\,.$$

If $\theta_0=+\infty$ consider the previous sequence $(m_n,\varepsilon_n)_{n\geq 0}$. One can in addition find a sequence $(\beta_n)_{n\geq 0}$ and some sequence $(\eta_n)_{n\geq 0}$ going to 0 such that for all (x,y),

$$\Lambda(m_n,\beta_n,x,y)-x\beta_n\varepsilon_n\leq\widetilde{J}(m)+\eta_n\,.$$

Since $\beta_n\in[0,\beta_\tau]$, we may assume that $\beta_n\to\beta$ up to considering a subsequence. β has to be positive, otherwise, taking limits as $n\to\infty$ we would get that for all (x,y)

$$\Lambda(m,0,x,y)=x+my\leq\widetilde{J}(m)<+\infty$$

which is impossible. Hence $\beta>0$ and taking limits again, we obtain $\Lambda(m,\beta,x,y)\leq\widetilde{J}(m)$ for some $\beta>0$ and all (x,y) so that $\bar{J}(m)\leq\widetilde{J}(m)$. $\square$

We turn to the converse inequality

Lemma 4.4. *Under the assumptions of Lemma 4.2, $\widetilde{J}\leq\bar{J}$.*

Proof. It is enough this time to assume that $\bar{J}(m)$ and thus $J(m)$ is finite. Notice furthermore than if $m=0$ and $\bar{J}(0)=\theta_0$ there is nothing to prove since $\widetilde{J}(0)\leq\liminf_{\varepsilon\to 0}\bar{J}^\varepsilon(0)\leq\theta_0$. As a consequence if $m=0$ we may assume in addition that $J(0)<\theta_0$.

Recall that $\bar{J}$ is defined in (3.5). Let μ_k be a minimizing sequence of $\bar{J}(m)$ in $\bar{\mathcal{M}}^1(\mathcal{X})$, i.e. $I(\mu_k) \leq \bar{J}(m) + \eta_k$ with $\eta_k \to_{k\to\infty} 0$ and $\mu_k(\varphi) = m$. From the definition of I, we have in particular $\mu_k(1/u) < +\infty$. Let us introduce μ_k^ε the push forward of μ_k by the application $t_\varepsilon : (u, w) \mapsto (u + \varepsilon, w)$ (i.e. if (τ, W) is distributed according to μ_k, μ_k^ε is the distribution of $(\tau + \varepsilon, W)$). Of course $\mu_k^\varepsilon(\mathcal{X}) \to_{\varepsilon\to 0} \mu_k(\mathcal{X})$ and $\mu_k^\varepsilon(1/u) \to_{\varepsilon\to 0} \mu_k(1/u)$ thanks to Lebesgue's bounded convergence theorem, and finally, since W is bounded for all considered measures, the same theorem shows that

$$\mu_k^\varepsilon(\varphi) = m_k^\varepsilon \to m = \mu_k(\varphi) \quad \text{as } \varepsilon \to 0.$$

Since the minimizing measure is not the null measure, we may assume that $\mu_k(\mathcal{X}) \geq \chi > 0$ for all k, so that $H(\bar{\mu}_k|\psi) < +\infty$.

In addition, we have for any bounded continuous function f

$$\begin{aligned}\int f(u, w)\, \bar{\mu}_k^\varepsilon(du, dw) &= \int f(u, w) \frac{1}{\mu_k^\varepsilon(1/u)} \frac{1}{u}\, \mu_k^\varepsilon(du, dw)\\ &= \int f(u + \varepsilon, w)\, \frac{1}{\mu_k(1/(u + \varepsilon))} \frac{1}{u + \varepsilon}\, \mu_k(du, dw)\\ &= \int f(u + \varepsilon, w)\, \frac{\mu_k(1/u)}{\mu_k(1/(u + \varepsilon))} \frac{u}{u + \varepsilon}\, \bar{\mu}_k(du, dw)\end{aligned}$$

Since $1/(u + \varepsilon) \leq 1/u$ which is μ_k integrable and $u/u + \varepsilon \leq 1$, it is thus immediately seen, thanks to Lebesgue's convergence theorem, that $\bar{\mu}_k^\varepsilon \to \bar{\mu}_k$ (and of course $\psi^\varepsilon \to \psi$) weakly as $\varepsilon \to 0$.

Since $H(\bar{\mu}_k|\psi) < +\infty$, $\bar{\mu}_k$ is absolutely continuous w.r.t. ψ with a density denoted by g_k. It follows that $\bar{\mu}_k^\varepsilon$ is absolutely continuous w.r.t. ψ^ε with a density given by

$$g_k^\varepsilon(u, w) = \frac{\mu_k(1/u)}{\mu_k(1/(u + \varepsilon))} \frac{u - \varepsilon}{u}\, g_k(u - \varepsilon, w) \;=\; C^\varepsilon\, \frac{u - \varepsilon}{u}\, g_k(u - \varepsilon, w)\,,$$

recall that $\psi^\varepsilon(u > \varepsilon) = 1$ so that we only need to consider such u's.

We thus have

$$\begin{aligned}H(\bar{\mu}_k^\varepsilon|\psi^\varepsilon) &= \int g_k^\varepsilon \,\log g_k^\varepsilon \; d\psi^\varepsilon\\ &= \int \log\left(C^\varepsilon\, \frac{u}{u + \varepsilon}\, g_k(u, w)\right)\, C^\varepsilon\, \frac{u}{u + \varepsilon}\, g_k(u, w)\, \psi(du, dw)\,.\end{aligned}$$

Notice that, for $\varepsilon \leq 1$, $C^\varepsilon\, \frac{u}{u+\varepsilon}\, g_k(u, w) \leq C^1\, g_k(u, w)$ so that

$$\begin{aligned}&\left|\log\left(C^\varepsilon\, \frac{u}{u + \varepsilon}\, g_k(u, w)\right)\, C^\varepsilon\, \frac{u}{u + \varepsilon}\, g_k(u, w)\right|\\ &\qquad \leq \max\left(e^{-1}\,;\; \log(C^1\, g_k(u, w))\, C^1\, g_k(u, w)\right)\end{aligned}$$

which is ψ integrable since $H(\bar{\mu}_k|\psi) < +\infty$. It follows, using again Lebesgue's theorem, that $\lim_{\varepsilon\to 0} H(\bar{\mu}_k^\varepsilon|\psi^\varepsilon) = H(\bar{\mu}_k|\psi)$.

For a given $\delta > 0$, we thus have

$$\liminf_{\varepsilon\to 0}\; \inf_{|z-m|<\delta}\; \bar{J}^\varepsilon(z) \leq \liminf_{\varepsilon\to 0}\; \bar{J}^\varepsilon(m_k^\varepsilon) \leq J(m) + \eta_k\,.$$

The upper bound does not depend on δ and it remains to make $\eta_k \to 0$ to get the result. $\square$

Step 3. In order to get the full LDP we need to check condition (3) in Theorem 2.2 i.e. that for all closed set F,

$$\inf_{z\in F} \bar{J}(z) \leq \limsup_{\varepsilon\to 0} \inf_{z\in F} \bar{J}^{\varepsilon}(z)\,.$$

We may of course assume that the right hand side is finite. For $\theta_0 < +\infty$ it is an immediate consequence of $\bar{J}(m) \leq \bar{J}^{\varepsilon}(m) + \beta_\tau\theta_0\,\varepsilon$.

If $\theta_0 = +\infty$, remark that for $\beta < \beta_\tau$

$$\sup_{x,y} \Lambda(m,\beta,x,y) \geq \Lambda(m,\beta,0,1) = m - \beta\log\mathbb{E}(e^{W}) \geq m - \beta K \ \geq m - \beta_\tau K\,,$$

and similarly

$$\sup_{x,y} \Lambda(m,\beta,x,y) \geq \Lambda(m,\beta,0,-1) = -m - \beta\log\mathbb{E}(e^{-W}) \geq -m - \beta_\tau K\,.$$

It follows $J^{\varepsilon}(m) \geq |m| - \beta_\tau K$ for all ε (including $\varepsilon = 0$), so that the level sets $\bar{J}^{\varepsilon} \leq M$ are all included in the ball $|m| \leq M + \beta_\tau K$.

For a closed set F, one can thus find a sequence ε_n, z_n with $\varepsilon \to_{n\to\infty} 0$ such that $\bar{J}^{\varepsilon_n}(z_n) \leq \inf_{z'\in F} J^{\varepsilon_n}(z') + 1/n$ and $z_n \in F \cap \{|m| \leq C\}$ for some C large enough. Taking a subsequence if necessary, we may assume that $z_n \to z \in F$ since F is closed. We have $\bar{J}^{\varepsilon_n}(z) \geq \bar{J}^{\varepsilon_n}(z_n) - (1/n)$. We can thus argue as in the proof of Lemma 4.3 to show that

$$\limsup_{n} \inf_{z'\in F} J^{\varepsilon_n}(z') = \limsup_{n} \bar{J}^{\varepsilon_n}(z) \geq \bar{J}(z) \geq \inf_{z'\in F} \bar{J}(z')\,. \quad \square$$

5. Deviations for the cumulative process in the general case. Proof of Theorem 2.4

We will now try to relax the boundedness assumption on W. We thus introduce $W^n = W \vee (-n) \wedge n$, ψ^n the distribution of (τ, W^n), I^n, $\bar{J}^n$ and J^n are defined accordingly. It is thus natural to look at

$$\widetilde{J}(m) = \sup_{\delta>0} \liminf_{n\to+\infty} \inf_{|z-m|<\delta} \bar{J}^n(z)\,. \tag{5.1}$$

We shall this time first compare $\widetilde{J}$ and $\bar{J}$.

Lemma 5.1. *It holds $\bar{J} \leq \widetilde{J}$.*

Proof. As in the proof of Lemma 4.3, $\sup_{x,y} \Lambda^n(m,\beta,x,y) \geq x_\tau + \beta$ so that if $J^n(m) < +\infty$ the infimum in β has to be taken for $\beta \leq J^n(m) - x_\tau$.

If $\widetilde{J}(m) < +\infty$ one can find a sequence $(m_n,\beta_n)_{n\geq 0}$ such that $m_n \to m$, $\beta_n \in (0,\beta_\tau]$ where $\beta_\tau \leq \widetilde{J}(m) + 1 - x_\tau$ and for n large enough and all (x,y),

$$x + m_n y - \beta_n \log\mathbb{E}(e^{x\tau + yW^n}) \leq \widetilde{J}(m) + 1/n\,.$$

Taking a subsequence if necessary we may assume that $\beta_n \to \beta_\infty$.

We want to pass to the limit in the previous inequality. We may assume that $\mathbb{E}(e^{x\tau}) < +\infty$, otherwise, for all $\beta > 0$,

$$x + my - \beta\log\mathbb{E}(e^{x\tau+yW}) = -\infty\,.$$

Since $e^{x\tau+yW^n}\mathbf{1}_{yW^n\leq 0} = e^{x\tau+yW^n}\mathbf{1}_{yW\leq 0}$ is dominated by $e^{x\tau}\,1_{yW\leq 0}$, which is assumed to be integrable, we may apply the bounded convergence theorem and get $\lim_n \mathbf{E}(e^{x\tau+yW^n}\mathbf{1}_{yW^n\leq 0}) =$

$\mathbb{E}(e^{x\tau+yW}\,\mathbf{1}_{yW\leq 0})$. The other part, $\lim_n \mathbf{E}(e^{x\tau+yW^n}\,\mathbf{1}_{yW^n>0}) = \mathbb{E}(e^{x\tau+yW}\,\mathbf{1}_{yW>0})$ is a consequence of the monotone convergence theorem.

We may thus conclude that for all (x, y),

$$x + my - \beta_\infty \log \mathbb{E}(e^{x\tau+yW}) \leq \widetilde{J}(m)\,,$$

hence $J(m) \leq \widetilde{J}(m)$, provided $\beta_\infty > 0$. If $\beta_\infty = 0$ we have obtained that for all x such that $\mathbb{E}(e^{x\tau}) < +\infty$, $x + my \leq \widetilde{J}(m)$ which is impossible if $m \neq 0$, or if $m = 0$ and $\widetilde{J}(0) < \theta_0$. Since $\bar{J}(0) \leq \theta_0$, the case $\widetilde{J}(0) \geq \theta_0$ is immediate. □

For the converse

Lemma 5.2. *It holds $\bar{J} \geq \widetilde{J}$.*

Proof. We shall follow the same route as for the proof of Lemma 4.4. We may similarly assume that $J(m)$ is finite and $J(0) < \theta_0$, so that the minimizing measure is not the null measure. We then consider a sequence μ_k such that $I(\mu_k) \leq \bar{J}(m) + \eta_k$, and we may assume again that $\mu_k(\mathcal{X}) \geq \chi > 0$ for all k so that $\sup_k H(\bar{\mu}_k|\psi) < +\infty$.

We may decompose ψ^n as

$$\psi^n(du, dw) = \mathbf{1}_{|w|<n}\psi(du, dw) + \gamma_+^n(du, dw) + \gamma_-^n(du, dw)$$

where γ_+^n is the joint law of $(\tau, n\,\mathbf{1}_{W\geq n})$ and γ_-^n is the joint law of $(\tau, -n\,\mathbf{1}_{W\leq -n})$. Of course ψ^n weakly converges towards ψ.

We now introduce $\mu_k^n = \mathbf{1}_{|w|<n}\,\mu_k$ so that

$$\bar{\mu}_k^n = \frac{\mu_k(1/u)}{\mu_k(\mathbf{1}_{|w|<n}\,(1/u))}\,\psi(|w| < n)\,\frac{d\bar{\mu}_k}{d\psi}\,\mathbf{1}_{|w|<n}\,\psi^n\,.$$

It is then easily seen that μ_k^n weakly converges to μ_k, that $\mu_k^n(\varphi) = m_k^n$ converges to $\mu_k(\varphi) = m$ and finally since $\mathbf{1}_{|w|<n}\,\psi^n = \mathbf{1}_{|w|<n}\,\psi$, denoting by

$$c_k^n = \frac{\mu_k(1/u)}{\mu_k(\mathbf{1}_{|w|<n}\,(1/u))}\,\psi(|w| < n)$$

that

$$H(\bar{\mu}_k^n|\psi^n) = \int c_k^n\,\log\left(c_k^n\,\frac{d\bar{\mu}_k}{d\psi}\right)\mathbf{1}_{|w|<n}\,d\bar{\mu}_k$$

goes to $H(\bar{\mu}_k|\psi)$ as n goes to infinity. We may thus conclude as in the proof of Lemma 4.4. □

In order to get an LDP result for $(Z_t/t)_{t\geq 0}$ it remains to study the approximation of $(Z_t/t)_{t\geq 0}$ by $\{(Z_t^n/t)_{t\geq 0}\}_{n\in\mathbb{N}}$. We may decompose

$$|Z_t - Z_t^n| = \sum_{i=1}^{M_t}(W_i - n)_+ + \sum_{i=1}^{M_t}(W_i + n)_-\,, \tag{5.2}$$

where $u_+ = \max(u, 0)$ and $u_- = \max(-u, 0)$. We then have

Lemma 5.3. *Assume that $\theta_0 > 0$ and $\eta_0 > 0$. For all $\delta > 0$,*

$$\lim_{n\to\infty}\,\limsup_{t\to\infty}\,\frac{1}{t}\,\log\mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| > 2\delta\right) \leq -\,\frac{\eta_0\,\delta}{2}\,.$$

In particular if $\eta_0 = +\infty$, $\{(Z_t^n/t)_{t\geq 0}\}_{n\in\mathbb{N}}$ is an exponentially good approximation of $(Z_t/t)_{t\geq 0}$.

Proof. Since η_0 and θ_0 are positive, $\mathbb{E}(\tau)$ and $\mathbb{E}(|W|)$ are both finite.

From (5.2), we deduce that

$$P\left(\left|\frac{Z_t}{t}-\frac{Z_t^n}{t}\right|>2\delta\right)\leq\mathbb{P}\left(\sum_{i=1}^{M_t}(W_i-n)_->\delta t\right)+\mathbb{P}\left(\sum_{i=1}^{M_t}(W_i-n)_+>\delta t\right)$$

Note that using the elementary $\log(a+b)\leq\max(\log(2a),\log(2b))$ it is enough to look at

$$\mathbb{P}\left(\sum_{i=1}^{M_t}(W_i-n)_+>\delta t\right),$$

since the other term can be treated similarly.

Using that the $(W_i)_{i\geq 1}$'s are i.i.d. we may write for $\delta>0$ and $c>0$, (as usual an empty sum is equal to 0 by convention)

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=1}^{M_t}(W_i-n)_+>\delta t\right)\\
&\leq\mathbb{P}\left(\sum_{i=1}^{\lfloor ct\rfloor}(W_i-n)_+>\frac{\delta t}{2}\right)+\mathbb{P}\left(\sum_{i=\lfloor ct\rfloor+1}^{M_t}(W_i-n)_+>\frac{\delta t}{2}\right)\\
&\leq\mathbb{P}\left(\sum_{i=1}^{\lfloor ct\rfloor}(W_i-n)_+>\frac{\delta t}{2}\right)\\
&\quad+\mathbb{P}\left(\left\{\sum_{i=\lfloor ct\rfloor+1}^{M_t}(W_i-n)_+>\frac{\delta t}{2}\right\}\cap\{1+\lfloor ct\rfloor\leq M_t<2\lfloor ct\rfloor\}\right)\\
&\quad+\mathbb{P}\left(\left\{\sum_{i=\lfloor ct\rfloor+1}^{M_t}(W_i-n)_+>\frac{\delta t}{2}\right\}\cap\{M_t\geq 2\lfloor ct\rfloor\}\right)\\
&\leq 2\mathbb{P}\left(\sum_{j=1}^{\lfloor ct\rfloor}(W_j-n)_+>\frac{\delta t}{2}\right)+\mathbb{P}\left(M_t\geq 2\lfloor ct\rfloor\right)
\end{aligned}$$

Study of $\mathbb{P}(M_t\geq 2\lfloor ct\rfloor)$. Start with the second term in the sum above. According to theorem 2.3 in [15], we know that M_t/t satisfies a LDP with rate function J_τ given by

$$J_\tau(u)=\begin{cases}\sup_{\lambda\in\mathbb{R}}\{\lambda-u\log\mathbb{E}(e^{\lambda\tau})\} & \text{if } u\geq 0,\\ \infty & \text{if } u<0.\end{cases}$$

Notice that $J_\tau(u)=u\,\Lambda^*(1/u,0)$ for $u>0$. In addition (see Lemma 2.6 in [15]) the supremum is achieved for $\lambda\leq 0$ if $u\in(1/\mathbb{E}(\tau),+\infty)$ and J_τ is non-decreasing on this interval.

It follows that for $2c>1/\mathbb{E}(\tau)$,

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(M_t\geq 2\lfloor ct\rfloor\right)\leq -J_\tau(\lfloor ct\rfloor).\tag{5.3}$$

In order to get $\lim_{n\to\infty}\limsup_{t\to+\infty}\frac{1}{t}\log\mathbb{P}(M_t\geq 2\lfloor ct\rfloor)\leq-\infty$ for some sequence c_n (to be chosen later) it remains to show that

$$J_\tau(u)\xrightarrow[u\to\infty]{}+\infty.$$

Recall that x_τ satisfies $\mathbb{E}(\mathrm{e}^{x_\tau \tau}) = \mathrm{e}^{-1}$, so that for $u \geq 0$,

$$J_\tau(u) = \sup_{\lambda\in\mathbb{R}}\{\lambda - u\log\mathbb{E}(\mathrm{e}^{\lambda\tau})\} \geq x_\tau - u\log\mathbb{E}(\mathrm{e}^{x_\tau\tau}) \geq u + x_\tau$$

yielding the desired result.

Study of $\mathbb{P}\left(\sum_{j=1}^{\lfloor ct\rfloor}(W_j - n)_+ > \frac{\delta t}{2}\right)$. We handle this term with Cramer's theorem. Defining

$$\Psi_n(\lambda) = \log\mathbb{E}\left[\mathrm{e}^{\lambda(W-n)_+}\right],$$
$$\Psi_n^*(x) = \sup_{\lambda\in\mathbb{R}}\{\lambda x - \Psi_n(\lambda)\},$$

we have

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\sum_{j=1}^{\lfloor ct\rfloor}(W_j-n)_+ > \delta t/2\right) &= \limsup_{t\to\infty}\frac{c}{\lfloor ct\rfloor}\log\mathbb{P}\left(\sum_{j=1}^{\lfloor ct\rfloor}(W_j-n)_+ > \delta t/2\right)\\ &\leq \limsup_{t\to\infty}\frac{c}{\lfloor ct\rfloor}\log\mathbb{P}\left(\sum_{j=1}^{\lfloor ct\rfloor}(W_j-n)_+ > \delta\lfloor ct\rfloor/2c\right)\\ &\leq -c\inf_{x\in[\delta/2c,+\infty)}\Psi_n^*(x).\end{aligned}$$

As the function $x \mapsto \Psi_n^*(x)$ is non-decreasing on $[\mathbb{E}((W-n)_+), +\infty)$, we have

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\sum_{j=1}^{\lfloor ct\rfloor}(W_j-n)_+ > \delta t/2\right) \leq -c\,\Psi_n^*(\delta/2c),$$

provided $\delta/2c \geq \mathbb{E}((W-n)_+)$. Notice that for $\lambda < \eta_0$,

$$c\,\Psi_n^*(\delta/2c) \geq \frac{\lambda\delta}{2} - c\log\left(1 + \mathbb{E}\left[(e^{\lambda(W-n)} - 1)\,\mathbb{1}_{W>n}\right]\right),$$

Since both $\mathbb{E}((W-n)_+)$ and $\log\left(1 + \mathbb{E}\left[(e^{\lambda(W-n)} - 1)\mathbb{1}_{W>n}\right]\right)$ are going to 0 as $n \to \infty$, it is always possible to choose c_n growing to infinity such that as $n \to \infty$

$$c_n\,\mathbb{E}((W-n)_+) \to 0 \quad\text{and}\quad c_n\log\left(1 + \mathbb{E}\left[(e^{\lambda(W-n)} - 1)\mathbb{1}_{W>n}\right]\right) \to 0,$$

We get

$$\lim_{n\to\infty}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\sum_{j=1}^{\lfloor ct\rfloor}(W_j-n)_+ > \delta t/2\right) \leq -\frac{\lambda\delta}{2}.$$

We may optimize in λ and plug the same sequence c_n in (5.3) completing the proof. $\square$

We will use the previous lemma to deduce

Corollary 5.4. *Under the assumptions of Lemma 5.3, $(Z_t/t)_{t\geq 0}$ is exponentially tight, i.e. for all $\alpha > 0$, there exists a compact set K_α such that*

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\frac{Z_t}{t} \notin K_\alpha^c\right) < -\alpha.$$

Proof. Since Z_t^n/t is an approximation of Z_t/t and satisfies a full LDP according to Theorem 4.1, we can decompose the probability as following: for each n, and for all δ:

$$\begin{aligned}\mathbb{P}\left(\frac{Z_t}{t} \notin [-A, A]\right) &\leq \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| > \delta\right) + \mathbb{P}\left(\frac{Z_t^n}{t} \notin [-A+\delta, A-\delta]\right) \\ &\leq \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| > \delta\right) + \mathbb{P}\left(\frac{Z_t^n}{t} < -A+\delta\right) + \mathbb{P}\left(\frac{Z_t^n}{t} > A-\delta\right). \\ &\leq 3\max\left(\mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| > \delta\right), \mathbb{P}\left(\frac{Z_t^n}{t} < -A+\delta\right), \right. \\ &\left. \mathbb{P}\left(\frac{Z_t^n}{t} > A-\delta\right)\right). \end{aligned} \tag{5.4}$$

By Lemma 5.3, Z_t^n/t and Z_t/t satisfy

$$\forall \delta > 0, \lim_{n\to\infty} \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| > \delta\right) = -\frac{\eta_0 \delta}{4},$$

i.e.

$$\forall \alpha > 0, \forall \delta > \frac{2\alpha}{\eta_0}, \exists n(\alpha,\delta), \forall n > n(\alpha,\delta), \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| > \delta\right) \leq -\alpha. \tag{5.5}$$

We just have to study $\mathbb{P}\left(\frac{Z_t^n}{t} > A-\delta\right)$ and the symmetric case. We know from Theorem 4.1 that:

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\frac{Z_t^n}{t} > B\right) \leq -\inf_{m>B} \bar{J}^n(m).$$

Since $\bar{J}^n$ has compact level sets, for all $\alpha > 0$ one can choose a level B_α such that $\forall m > B_\alpha$, $J^n(m) > \alpha$. The result follows by choosing $A = B_\alpha + \delta$. □

Proof of Theorem 2.4. In the case where $\eta_0 = +\infty$, using the approximation W^n, Lemmas 5.1 and 5.2 allow to obtain the weak LDP. The full LDP derives from Corollary 5.4 combined with Lemma 2.3.

If $\eta_0 < +\infty$ we only obtain asymptotic deviation bounds. Recall that $m = \mathbb{E}(W)/\mathbb{E}(\tau)$ is the limit of Z_t/t as $t \to +\infty$. For all $\kappa \in (0, 1)$ and $a > 0$, it holds

$$\mathbb{P}\left(\frac{Z_t}{t} \geq m+a\right) \leq \mathbb{P}\left(\frac{Z_t^n}{t} \geq m+\kappa a\right) + \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| \geq (1-\kappa)a\right),$$

so that, for all $n \geq 0$,

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\frac{Z_t}{t} \geq m+a\right) \\ &\quad \leq \max\left[\limsup_{t\to+\infty} \frac{1}{t} \log \mathbb{P}\left(\frac{Z_t^n}{t} \geq m+\kappa a\right)\right. \\ &\quad \left. ; \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\left|\frac{Z_t}{t} - \frac{Z_t^n}{t}\right| \geq (1-\kappa)a\right)\right].\end{aligned}$$

Taking the lim inf in n we deduce

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\frac{Z_t}{t}\geq m+a\right)\leq \max\left[\liminf_{n\to\infty}(-\inf_{z\geq m+\kappa a}\bar{J}^n(z))\,;\,-\frac{\eta_0\,(1-\kappa)a}{4}\right]$$
$$\leq -\min\left[\limsup_{n\to\infty}(\inf_{z\geq m+\kappa a}\bar{J}^n(z))\,;\,\frac{\eta_0\,(1-\kappa)a}{4}\right].$$

To complete the proof of the Theorem it is enough to prove

Lemma 5.5. *Assume $\eta_0>\infty$, then for any $z_0\in\mathbb{R}$,*

$$\limsup_{n\to\infty}(\inf_{z\geq z_0}\bar{J}^n(z))\geq \inf_{z\geq z_0}\bar{J}(z).$$

Proof. The proof is close to the one of Lemma 5.1. We may of course assume that the left hand side is finite, denoted by $C(z_0)$. As usual, for a fixed $\varepsilon>0$, we may find a sequence $(z_n)_{n\geq 0}$ such that for any $n\in\mathbb{N}$, $z_n\geq z_0$ and $\inf_{z\geq z_0}\bar{J}^n(z)+\varepsilon\geq\bar{J}^n(z_n)$, so that $\limsup_{n\to\infty}\bar{J}^n(z_n)\leq C(z_0)+\varepsilon$.

We want to show that the sequence $(z_n)_{n\geq 0}$ is bounded. The key point is to remark that, taking the sign of y into account

$$x+zy-\beta\log\mathbb{E}\left(e^{x\tau+yW^n}\right)\geq x+zy-\beta\log\mathbb{E}\left(e^{x\tau+|y||W^n|}\right)$$
$$\geq x+zy-\beta\log\mathbb{E}\left(e^{x\tau+|y||W|}\right)$$

so that for all n,

$$\bar{J}^n(z)\geq J^{|.|}(z):=\inf_{\beta>0}\sup_{x\in\mathbb{R},y\geq 0}\left\{x+|z|y-\beta\log\mathbb{E}\left(e^{x\tau+y|W|}\right)\right\}.$$

As before, taking $y=0$ we see that the infimum in β has to be taken for $\beta\leq\beta_\tau=C(z_0)+1-x_\tau$, at least for n large enough.

Taking $x=0$ we see that $J^{|.|}(z)\leq C(z_0)+\varepsilon$ implies

$$|z|(\eta_0/2)\leq C(z_0)+\beta_\tau\,\log\mathbb{E}\left(e^{(\eta_0/2)|W|}\right),$$

i.e $|z|\leq A$ for some positive A that does not depend on n. This shows that $(z_n)_{n\geq 0}$ is bounded, so that taking a subsequence if necessary $z_n\to z_{lim}\geq z_0$.

Consider $\bar{J}(z_{lim})$. We may now mimic the proof of Lemma 5.1 replacing m_n by z_n and m by z_{lim}, so that

$$\inf_{z\geq z_0}\bar{J}(z)\leq\bar{J}(z_{lim})\leq C(z_0)+\varepsilon.$$

It remains to let ε go to 0. □

6. Application to Hawkes processes. Corrigendum

In [6] Theorem 2.12 and Corollary 2.13, we gave an application to Hawkes processes of our main results, with a *wrong* bound.

As we have seen the correct one in Theorem 2.12 is $(1-\kappa)\theta_0 a/4$ (θ_0 there is η_0 in the present paper), the factor 1/4 is missing in [6]. The correct term in Corollary 2.13 is also $(1-\kappa)\theta_0 a/4$. Indeed according to Eq. (2.9) therein, $N_t^h=\hat{N}_t^h+R_t^h$ with $0\leq R_t^h\leq W_{M_t^h+1}$. If W is bounded we may thus write $N_t^h=\mu_t^\varepsilon(\varphi)+A_t^\varepsilon$ where $A_t^\varepsilon\leq\frac{K}{t}((M_t-M_t^\varepsilon)+2)$, so that the proof of Theorem 4.1 remains valid replacing Z_t/t by N_t^h/t.

Also remark that we have to replace J by $\bar{J}$, i.e. take care of the case $z=0$, even if here $m>0$ since $W\geq 0$ and $W\neq 0$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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