Asymptotic theorems for cumulative processes

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IMT

27th October 2021,
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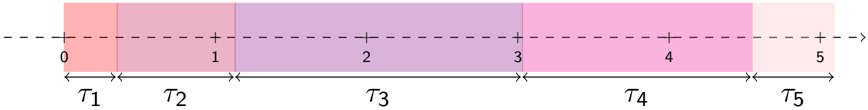
Renewal process

Definition

Let $(\tau_i)_{i\in\mathbb{N}^*}$ an i.i.d. sequence of random variable, such that $\tau_i > 0$ a.s.
Then $S_n = \sum_{i=1}^n \tau_i$ is a renewal process. The counting process associated to S_n is

$$M_t = \sup_{n\in\mathbb{N}} \left\{ \sum_{i=1}^{n} \tau_i \leq t \right\}.$$



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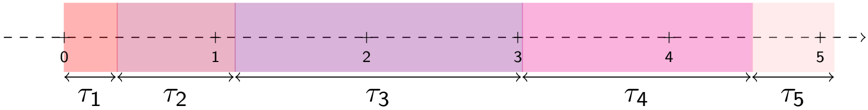


Example: Poisson process

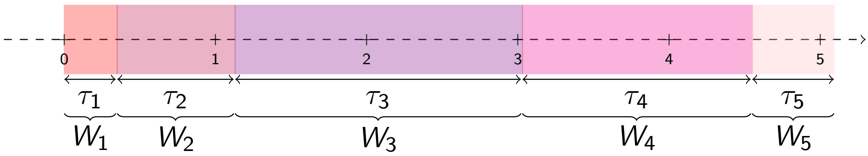
If $\tau_i \sim \mathcal{E}(\lambda)$, then M_t is a Poisson process of parameter λ.

Cumulative process

Definition

Let $(\tau_i, W_i)_i$ i.i.d. couples of random variable.
Let M_t the counting process associated with $(\tau_i)_i$: $M_t = \sup_{n\in\mathbb{N}} \{\sum_{i=1}^{n} \tau_i \leq t\}$. The *cumulative process* associated with $(\tau_i, W_i)_i$ is

$$Z_t = \sum_{i=1}^{M_t} W_i.$$



Law of large numbers and TCL

Proposition

Assume $\mathbb{E}[W] < \infty$ and $\mathbb{E}[\tau] < \infty$. Let $m = \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]}$. Then

$$\frac{Z_t}{t} \xrightarrow[t\to\infty]{a.s.} m.$$

Moreover, if $Var[W] < \infty$ and $Var[\tau] < \infty$, then

$$\sqrt{t}\left(\frac{Z_t}{t} - m\right) \xrightarrow[t\to\infty]{law} \mathcal{N}\left(0, \sigma^2\right),$$

where $\sigma^2 = \frac{Var(W - m\tau)}{\mathbb{E}(\tau)}$.

Idea

Idea for Law of Large Numbers

For renewal processes, we have

$$\frac{M_t}{t} \xrightarrow[t\to\infty]{a.s.} \frac{1}{\mathbb{E}[\tau]}.$$

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$$\frac{M_t}{t} \xrightarrow[t\to\infty]{a.s.} \frac{1}{\mathbb{E}[\tau]} \text{ so } \frac{Z_t}{t} = \frac{M_t}{t}\left(\frac{1}{M_t}\sum_{i=1}^{M_t} W_i\right).$$

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$$\frac{M_t}{t} \xrightarrow[t\to\infty]{a.s.} \frac{1}{\mathbb{E}[\tau]} \text{ so } \frac{Z_t}{t} = \underbrace{\frac{M_t}{t}}_{\xrightarrow[t\to\infty]{a.s.} \frac{1}{\mathbb{E}[\tau]}} \underbrace{\left(\frac{1}{M_t} \sum_{i=1}^{M_t} W_i \right)}_{\xrightarrow[t\to\infty]{a.s.} \mathbb{E}[W]}.$$

LLN proved.

Idea

Idea for LCT

$$\frac{Z_t}{t} - m = \frac{\sum_{i=1}^{M_t} W_i - tm}{t} = \frac{\sum_{i=1}^{M_t}(W_i - \tau_i m) + \left(\sum_{i=1}^{M_t} \tau_i m - tm\right)}{t}$$

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$$\begin{aligned}\sqrt{t}\left(\frac{Z_t}{t} - m\right) &= \sqrt{t}\frac{\sum_{i=1}^{M_t}(W_i - \tau_i m)}{t} + \sqrt{t}\frac{\sum_{i=1}^{M_t} \tau_i m - tm}{t} \\ &= \frac{\sqrt{M_t}}{\sqrt{t}}\frac{\sum_{i=1}^{M_t}(W_i - \tau_i m)}{\sqrt{M_t}} + \sqrt{t}m\frac{\sum_{i=1}^{M_t} \tau_i - t}{t}\end{aligned}$$

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$$\begin{aligned}
\sqrt{t}\left(\frac{Z_t}{t} - m\right) &= \sqrt{t}\frac{\sum_{i=1}^{M_t}(W_i - \tau_i m)}{t} + \sqrt{t}\frac{\sum_{i=1}^{M_t} \tau_i m - tm}{t} \\
&= \underbrace{\frac{\sqrt{M_t}}{\sqrt{t}}}_{\underset{t\to\infty}{\longrightarrow} \frac{1}{\sqrt{\mathbb{E}[\tau]}}} \underbrace{\frac{\sum_{i=1}^{M_t}(W_i - \tau_i m)}{\sqrt{M_t}}}_{\underset{t\to\infty}{\longrightarrow} \mathcal{N}} + \underbrace{\sqrt{t}m\frac{\sum_{i=1}^{M_t} \tau_i - t}{t}}_{\underset{t\to\infty}{\longrightarrow} 0}
\end{aligned}$$

Large deviation principle for Cumulative Process

Important assumptions

- $\exists \beta_0 \in (0, +\infty]$ such that $\mathbb{E}[e^{\beta\tau}] < \infty$ for $\beta < \beta_0$,

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- $\exists \beta_0 \in (0, +\infty]$ such that $\mathbb{E}[e^{\beta\tau}] < \infty$ for $\beta < \beta_0$,
- $\exists \theta_0 \in (0, +\infty]$ such that $\mathbb{E}[e^{\theta|W|}] < \infty$, for $\theta < \theta_0$,
- (other assumption : for all interval $\mathcal{I}$ such that $\mathbb{P}(W \in \mathcal{I}) > 0$, it holds : for all $t \geq 0$, $\mathbb{P}(\tau > t\,,\ W \in \mathcal{I}) > 0$)

Large deviation principle for Cumulative Process

Rate functions

For W^n a well-chosen reduction of W, we introduce the Cramer transform for $(a, b) \in \mathbb{R}^2$, and the rate function J^n associated for $z \in \mathbb{R}^+$

$$\Lambda_n^*(a, b) = \sup_{x,y} \left\{ ax + by - \ln \mathbb{E}\left(e^{x\tau + yW^n}\right)\right\} \quad \text{and} \quad J^n(z) = \inf_{\beta > 0} \beta \Lambda_n^* \left(\frac{1}{\beta}, \frac{z}{\beta}\right).$$

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We also define

$$\tilde{J}(z) = \sup_{\delta > 0} \liminf_{n \to \infty} \inf_{|y - z| < \delta} J^n(y).$$

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We also define

$$\tilde{J}(z) = \sup_{\delta>0} \liminf_{n\to\infty} \inf_{|y-z|<\delta} J^n(y).$$

For W, we introduce the Cramer transform for $(a,b) \in \mathbb{R}^2$, and the rate function J associated for $z \in \mathbb{R}^+$

$$\Lambda^*(a,b) = \sup_{x,y} \left\{ ax + by - \ln \left(\mathbb{E}\left[e^{x\tau + yW} \right] \right) \right\} \quad \text{and} \quad J(z) = \inf_{\beta>0} \beta \Lambda^* \left(\frac{1}{\beta}, \frac{z}{\beta} \right).$$

Large deviation principle for Cumulative Process

Theorem

- If $\theta_0 = +\infty$ (in particular if W is bounded) then $\frac{1}{t}\sum_{i=1}^{M_t} W_i$ satisfies a full LDP with good rate function $\tilde{J}$, i.e.

$$\text{for any closed set } \mathcal{C} \in \mathbb{R}, \qquad \limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{1}{t}\sum_{i=1}^{M_t} W_i \in \mathcal{C}\right) \leq -\inf_{m\in\mathcal{C}} \tilde{J}(m),$$

$$\text{for any open set } \mathcal{O} \in \mathbb{R}, \qquad \liminf_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{1}{t}\sum_{i=1}^{M_t} W_i \in \mathcal{O}\right) \geq -\inf_{m\in\mathcal{O}} \tilde{J}(m).$$

Large deviation principle for Cumulative Process

Theorem

- If $\theta_0 = +\infty$ (in particular if W is bounded) then $\frac{1}{t}\sum_{i=1}^{M_t} W_i$ satisfies a full LDP with good rate function $\tilde{J}$. We also have the following inequalities

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{1}{t}\sum_{i=1}^{M_t} W_i \geq m + a\right) \leq - \inf_{z\geq m+a} J(z),$$

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{1}{t}\sum_{i=1}^{M_t} W_i \leq m - a\right) \leq - \inf_{z\leq m-a} J(z).$$

Large deviation principle for Cumulative Process

Theorem

- If $\theta_0 = +\infty$ (in particular if W is bounded) then $\frac{1}{t}\sum_{i=1}^{M_t} W_i$ satisfies a full LDP with good rate function $\tilde{J}$.
- If $\theta_0 < +\infty$, denoting $m = \mathbb{E}(W)/\mathbb{E}(\tau)$ we have for all $a > 0$

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{1}{t}\sum_{i=1}^{M_t} W_i \geq m + a\right) \leq -\min\left[\inf_{z\geq m+(a/2)} J(z) \ , \ \theta_0 a/4\right],$$

and

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{1}{t}\sum_{i=1}^{M_t} W_i < m - a\right) \leq -\min\left[\inf_{z\leq m-(a/2)} J(z) \ , \ \theta_0 a/4\right].$$

Conclusion

We have:

- Law of large numbers
- Central limit theorem
- Large deviation principle : every exponential moment of W
  Deviations inequalities.

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- Law of large numbers
- Central limit theorem
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  Deviations inequalities.

Leads

- Obtain finite properties on cumulative process (finite deviations, etc)

Bibliography

[1] Søren Asmussen. *Applied probability and queues*. Second. Vol. 51. Applications of Mathematics (New York). Stochastic Modelling and Applied Probability. Springer-Verlag, New York, 2003, pp. xii+438.

[2] Patrick Cattiaux, L. C., and Manon Costa. *Large deviation principles for cumulative processes and applications*. 2021. arXiv: 2109.07800.